Efficient Computation of Change-Graph Scores

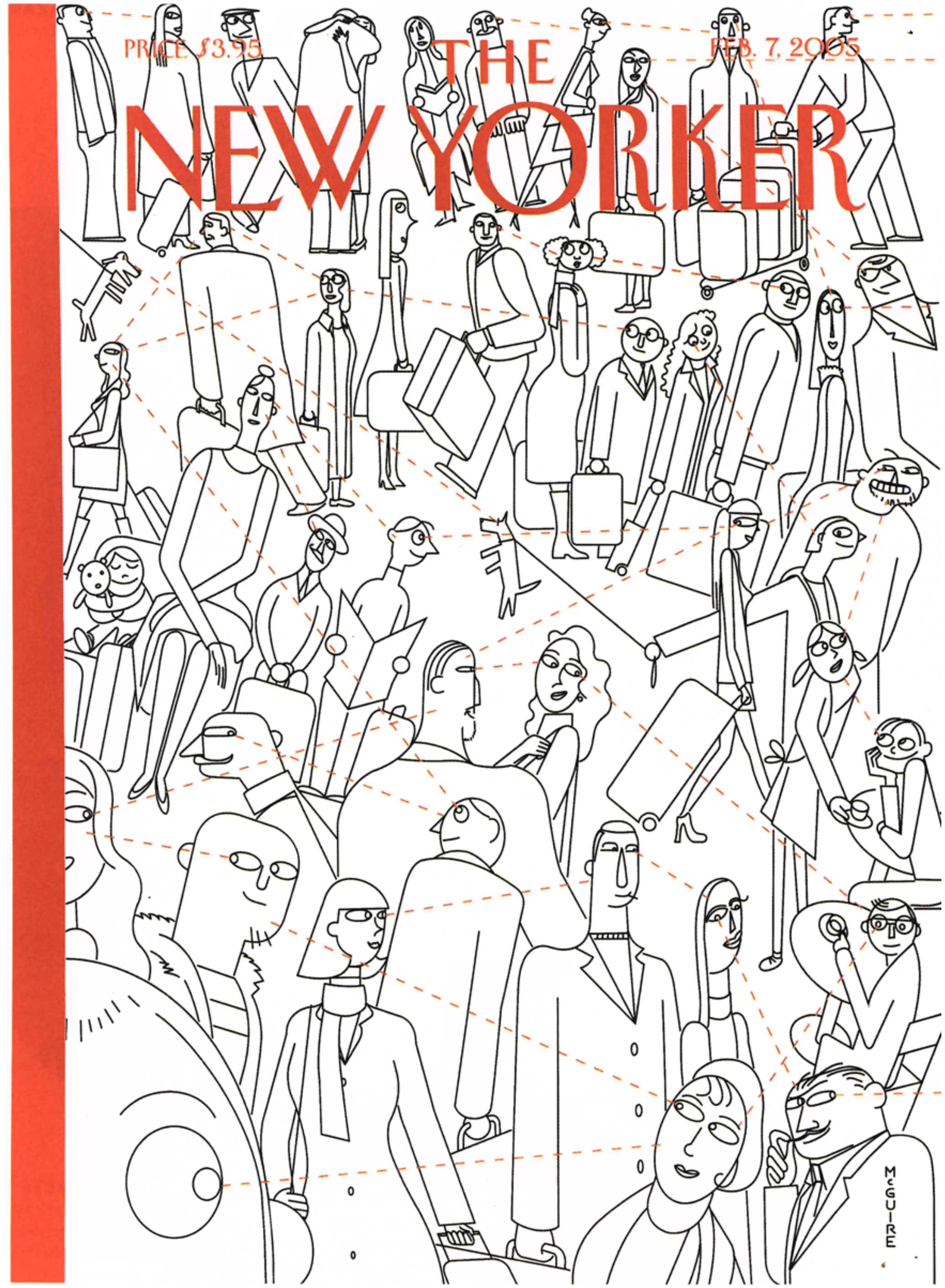
David Eppstein

(includes joint work with Emma Spiro, Mike Goodrich, Darren Strash, Lowell Trott, and Maarten Löffler)

Context: analysis of social networks

Represent interactions among people and their environments as graphs

(often: vertices = people,
edges = pairwise interactions)



Goals:

Predict human behavior

Detect anomalous behavior

Handle varied types of graph data and scale well to large networks

Mathematical modeling of social networks

Develop mathematical models
with a small number of meaningful numerical parameters
that generate graphs resembling real social networks

Not a pipe, but a model of a pipe

René Magritte, *The Treachery of Images*, 1928–9

Why?

- Fitting the parameters to real data tells us how real social nets behave

- The parts of the real networks that do not match the model may be anomalous

- We can use the model to generate test data for other analysis algorithms

Exponential random graph model: graphs shaped by their local structures

Define local features that may be present in a graph:

- Presence of an edge
- Degree of a vertex
- Small subgraphs

Assign weights to features: positive = more likely, negative = less likely

Log-likelihood of G = sum of weights of features + normalizing constant

Different feature sets and weights give different models capable of fitting different types of social network

Public-domain image by Mohylek on Wikimedia commons, http://commons.wikimedia.org/wiki/File:Magnifying.jpg

Probabilistic reasoning in exponential random graphs

Most basic problem: pull the handle, generate a random graph from the model

With a generation subroutine, we can also:

- Find normalizing constant
- Fit weights to data
- Understand typical behavior of graphs in this model (e.g. how many edges?)
- Detect unusual structures in real-world graphs

Standard method for random generation: Markov Chain Monte Carlo (random walk)

Start with any graph

Repeatedly choose a random edge to add or remove

Calculate change to log-likelihood

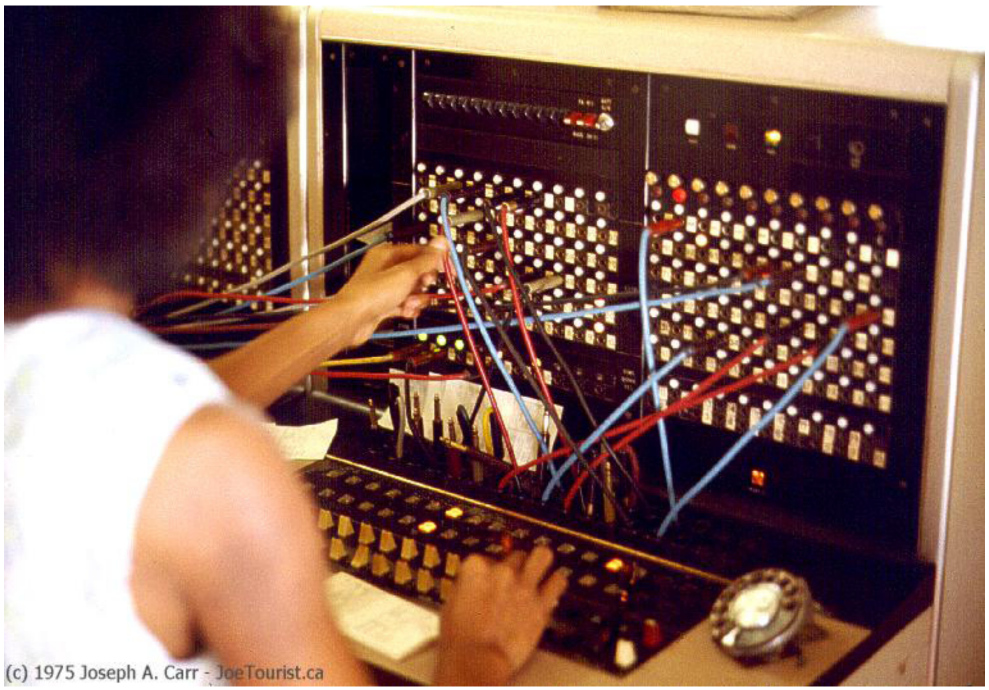
Choose whether to perform the update
(positive change score: always perform
negative change score: sometimes reject)

After enough steps, graph is random with correct probability distribution

The key algorithmic subproblem:

Add and remove edges in a dynamic graph

At each step, update feature counts
(how many of each type of small subgraph it has)



A telephone switchboard, an early example of a dynamic graph

Photo by Joseph A. Carr, 1975, available online under a free license at http://commons.wikimedia.org/wiki/File:JT_Switchboard_770x540.jpg

Because this is in the inner loop, it must be very fast

MURI-funded work on this problem:

The *h*-index of a graph and its application to dynamic subgraph statistics (with E. S. Spiro)
Presented at WADS, Banff, Canada, 2009.
Lecture Notes in Comp. Sci. 5664, 2009, pp. 278-289.

> Shortlisted for best paper award.
> Undirected graphs, feature = subgraph with ≤ 3 vertices

Extended dynamic subgraph statistics using *h*-index parameterized data structures
(with M. T. Goodrich, D. Strash, and L. Trott)
in preparation

> Directed graphs, larger numbers of vertices per feature
> See poster session

New research still under development (with M. T. Goodrich, M. Löffler)

> Geometric graphs and geometric features

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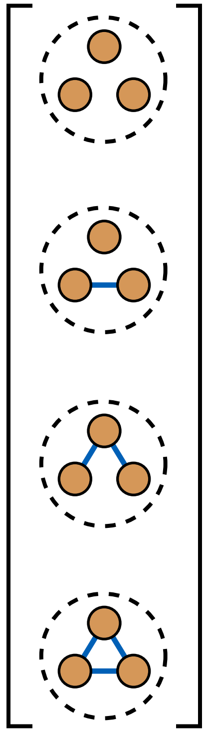
Interdependence among 3-vertex feature counts

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$= \begin{bmatrix} n(n-1)(n-2)/6 \\ m(n-2) \\ \Sigma \deg(v)\,(\deg(v)-1)/2 \\ \text{number of triangles} \end{bmatrix}$$

So if we can maintain the number of triangles in a dynamic graph
we can easily compute all other counts

Degree-based partitioning of a graph

Select a number D

Partition vertices into two subsets:

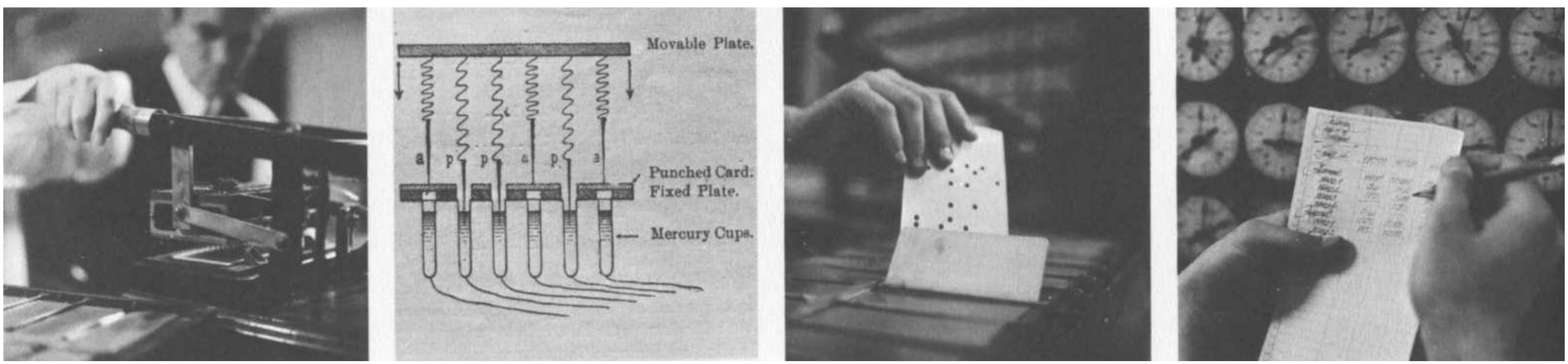
L: many vertices with degree less than D
H: few vertices with degree greater than D

Boys choosing sides for hockey on Sarnia Bay, Ontario, December 29, 1908. Public domain image from Library and Archives Canada / John Boyd Collection / PA-060732
http://www.collectionscanada.gc.ca/hockey/024002-2300-e.html

What we store:
Number of paths through low-degree vertices

Maintain hash table C indexed by pairs (*u*,*v*) of vertices

C[u,v] = number of two-edge paths u—L—v



Hollerith 1890 census tabulator from http://www.columbia.edu/acis/history/census-tabulator.html

When edge (u,v) is added or removed:

The number of triangles with the third vertex in L is stored in C[u,v]
(look it up there)

The number of triangles with a third vertex w in H
can be counted by examining all possibilities for w
(loop over all vertices in H and test whether each one forms a triangle)

If u belongs to L, add degree(v) to C[u,w] for each neighbor w of u
(perform a symmetric update if v belongs to L)

(Very infrequently) update the partition into low and high degree

How much time does it take per change?

Finding triangles involving changed edge takes **O(|H|)**

Each edge is involved in O(D) x—L—x paths, so updating hash table after a change takes **O(D)**

If L/H partition ever changes, update counts for all x—L—x paths through moved vertex taking time **$O(D^2)$**

How to choose D so |H| + D is small and partition changes infrequently?

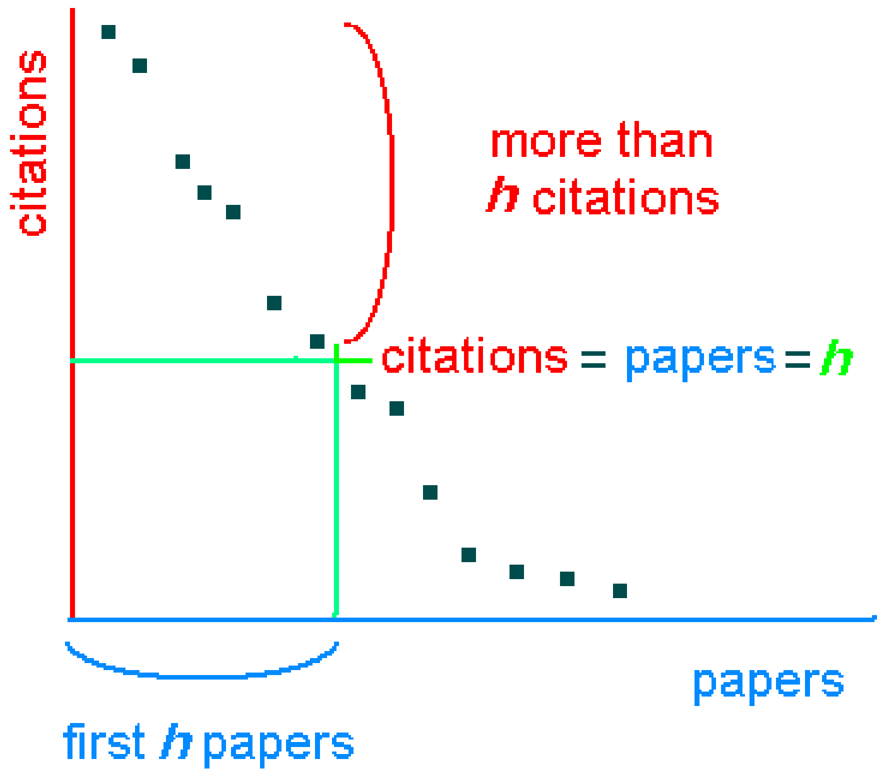
A detour into bibliometrics

How to measure productivity of an academic researcher?

Total publication count: encourages many low-impact papers

Total citation count: unduly influenced by few high-impact pubs

h-index [J. E. Hirsch, PNAS 2005]:
maximum number such that *h* papers each have ≥ *h* citations



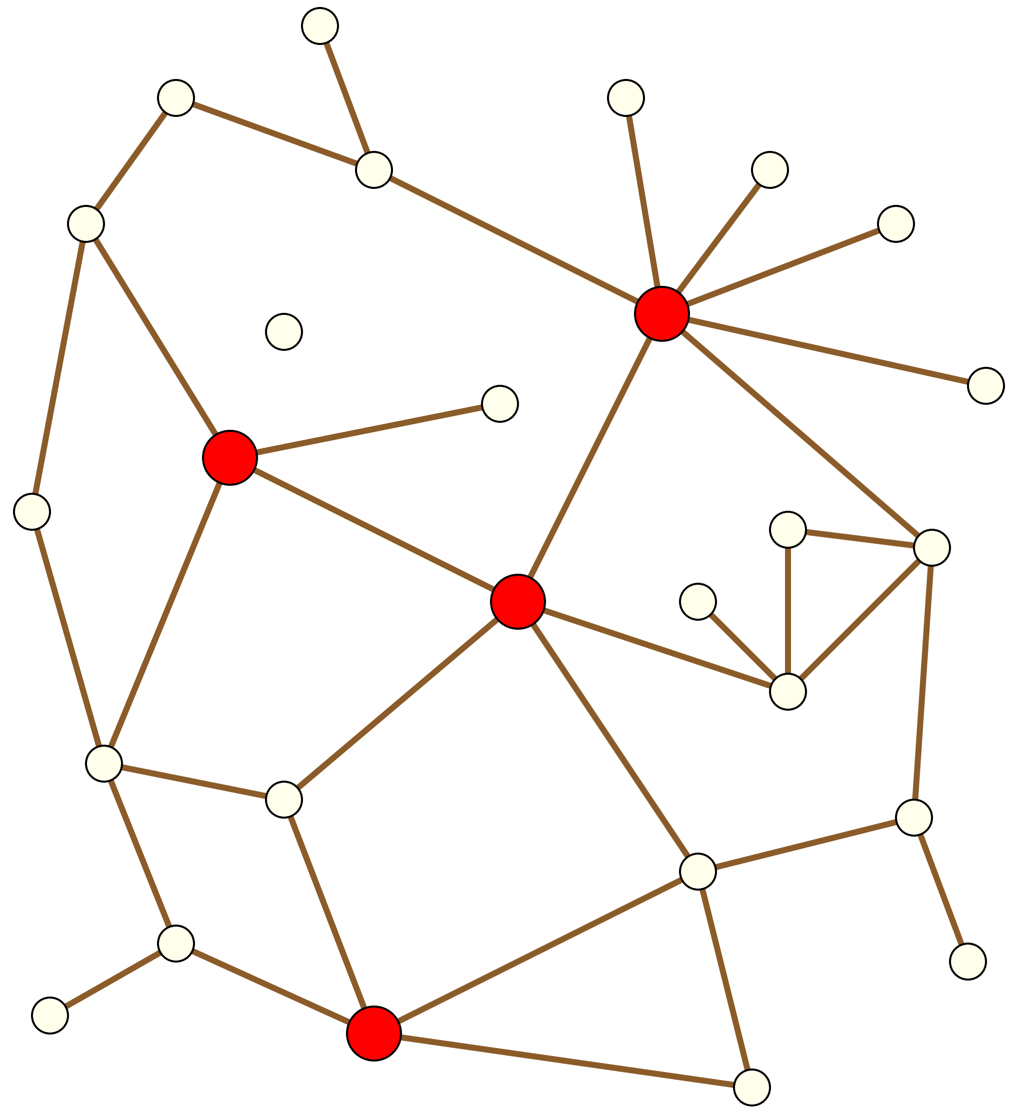
The *h*-index of a graph:

Maximum number such that
h vertices each have ≥ *h* neighbors

H = set of *h* high-degree vertices
L = remaining vertices, degree ≤ *h*

Provides optimal tradeoff
between |H| and D

Never more than sqrt(*m*)
Else H would have too many edges



Results:

We can maintain the *h*-index of a dynamic graph
in constant time per update
(details beyond the scope of this talk)

A relaxed degree partition based on the *h*-index changes very rarely
On average, some vertex changes sides once in every O(h) updates

As a consequence, we can maintain triangle counts and change scores
in time O(h) per update

All algorithms are simple and implementable

Later work (Trott poster) generalizes this to more complex features

Still need to do: implement them and test their actual performance


WORLD'S END
Please drive carefully