Decision Theoretic Foundations for Statistical Network Models

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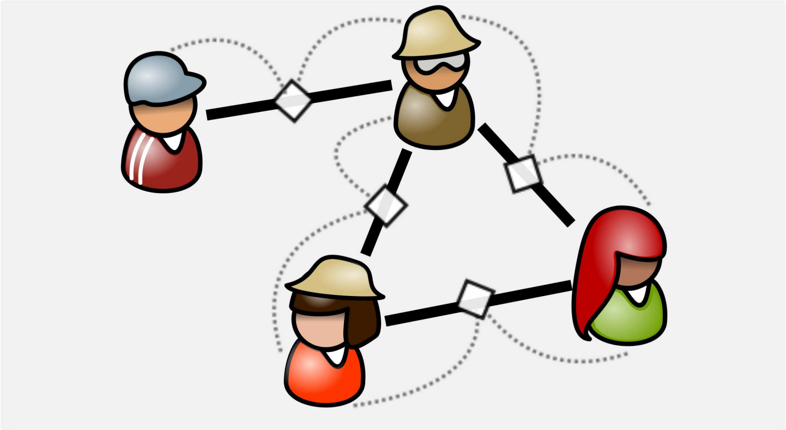
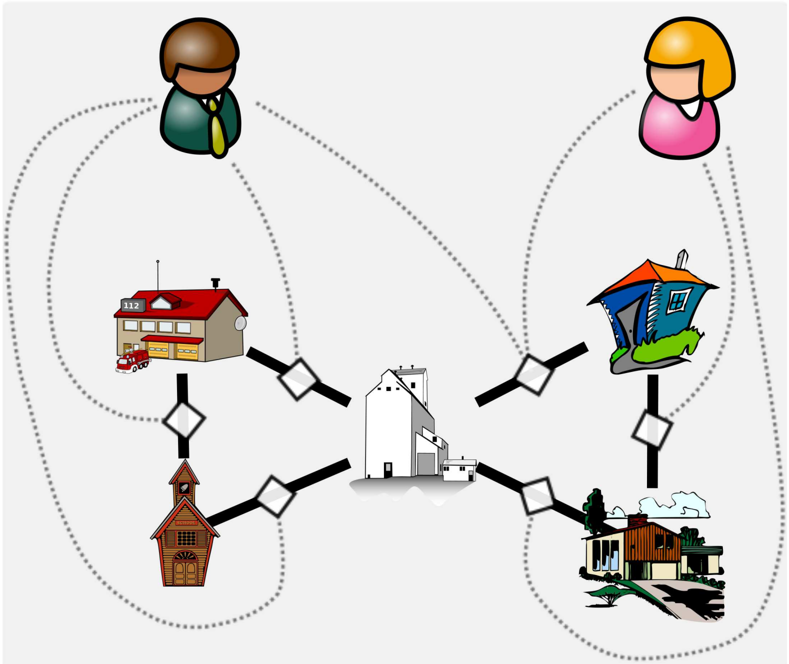
Problem: Interpreting Cross-sectional Network Models

- Tremendous progress in recent decades on cross-sectional network models Robins and Morris (2007); Wasserman and Robins (2005)
- Powerful, but often difficult to interpret; no general way to relate to agent behavior
  - Goodreau et al. (2008) make an attempt, but lack formal justification; Snijders (2001) provides at least one special case (mostly ignored)
- Some success in dynamic modeling area (e.g., Snijders (1996; 2005)) but dynamic data *much* harder to obtain
  - Also, growing agent-based and game theoretic literature (see e.g., Jackson (2006)), but no general link to inference
- Question: Can we produce a behaviorally reasonable micro-foundation for (at least some) cross-sectional network models?
  - Should be based on a behaviorally credible decision process
  - Should allow deduction of equilibrium network behavior
  - Should (at least sometimes) allow inference for actor preferences given observed structure
- Answer: *Yes, we can!* (In many cases, at least.)

Choosing Your Friends – or Others'

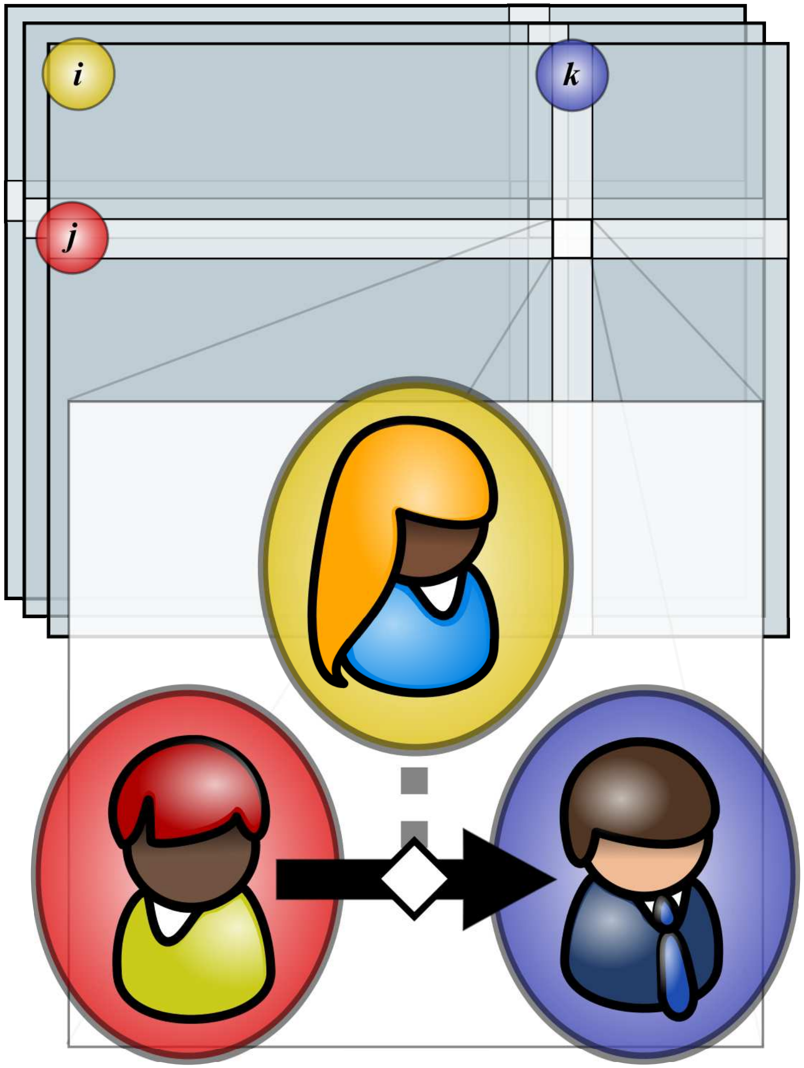
- Assume a set of N agents, A, whose actions jointly determine a network on n vertices with adjacency matrix $Y \in \mathcal{Y}_n$
  - Not required that $A = V$; agents may or may not be vertices (e.g., in designed networks)
  - Y is *manifest relation*, over which agents have preferences
  - Y can be directed/undirected, hypergraphic, etc. (but we treat as dyadic here)





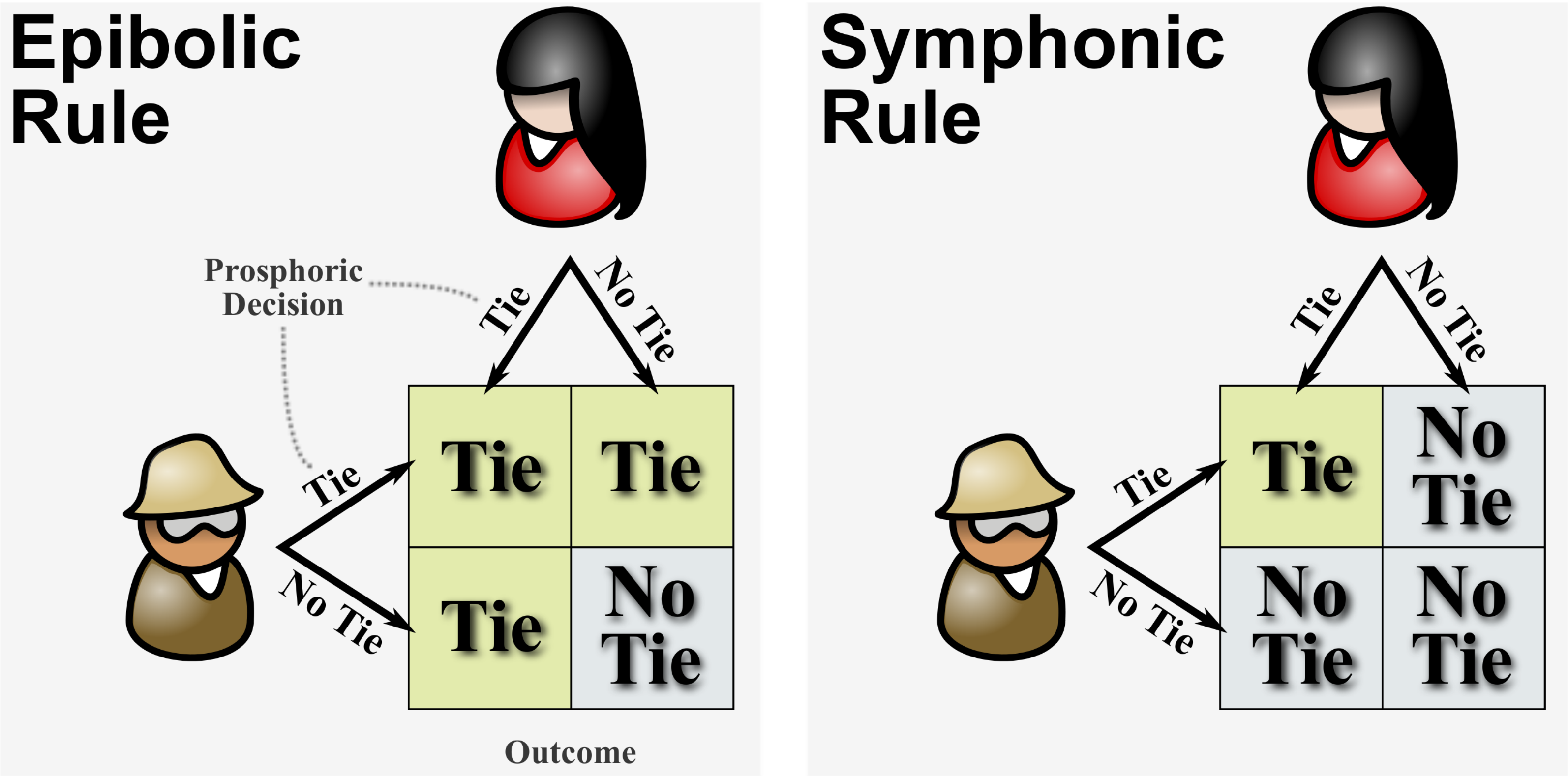
Resolving Relationships

- From choices to outcomes: the *prosphoric array*
  - Let $c_{ij} \subseteq A$ be the minimum lexically ordered ℓ-tuple of agents whose behaviors determine Y_{ij}
  - Let $P \in \mathcal{P}_N$ be an $\ell \times N \times N$ array, w/P_{ijk} recording the choice of ith agent of c_{jk} about Y_{ij}
  - *Resolution function* $r : \mathcal{P}_N \mapsto \mathcal{Y}_n$ maps individual choices to manifest relations
    - c_{jki} need not be v_i or v_j (but often will be)
    - Agents choose outcomes directly only when $\ell = 1$ (*unilateral* control); otherwise, relationship is *multilateral*



Bilateral Resolution Functions

- *Bilateral relationships* an important special case; include most undirected social ties
- Two common types (with natural generalizations for $\ell > 2$):
  - *Epibolic* – either party can impose the tie upon the other
  - *Symphonic* – either party can prevent/sever the tie



The Decision Model

- Agents choose the elements of P they control, under the following assumptions:
  - Decisions are instantaneous and element-wise
  - Decisions are myopic, and treat other elements of P as being fixed
  - Agent utilities, u, are functions of $r(P) = Y$ (and possibly covariates)
  - Decisions are made using a logistic choice process (McFadden, 1973)
- Consider a hypothetical move from state $P^{(i-1)}$ to P^{i}, in which agent a evaluates the k, l edge (a being the jth controller for that edge in P). Then the chance of a's selecting $P^{i}_{jkl} = 1$ is given by

$$\Pr\left(P^{(i)}_{jkl} = \left(p^{(i-1)}\right)^{+}_{jkl} \middle| \left(P^{(i-1)}\right)^{c}_{jkl} = \left(p^{(i-1)}\right)^{c}_{jkl}, u_a\right)$$

$$= \operatorname{logit}^{-1}\left[u_a\left(r\left(\left(p^{(i-1)}\right)^{+}_{jkl}\right)\right) - u_a\left(r\left(\left(p^{(i-1)}\right)^{-}_{jkl}\right)\right)\right] \quad (1)$$

  - P^{c}_{ijk} indicates all elements of P other than the i, j, kth
  - P^{+}_{ijk} indicates P^{c}_{ijk} with $P_{ijk} = 1$
  - P^{-}_{ijk} indicates P^{c}_{ijk} with $P_{ijk} = 0$
  - u_a is the utility function of agent a

The Utility Function

- We have already said that u_a is a function of Y (via P)
- Particularly important case drawn from theory of *potential games*
  - General defn: Let X by a strategy set, u a vector utility functions, and A a set of players. Then (A, X, u) is said to be a *potential game* if $\exists\, \rho : X \mapsto \mathbb{R}$ such that, for all $i \in A$, $u_i\left(x'_i, x_{-i}\right) - u_i\left(x_i, x_{-i}\right) = \rho\left(x'_i, x_{-i}\right) - \rho\left(x_i, x_{-i}\right)$ for all $x, x' \in X$.
  - Our case: assume exists a *potential function* $\rho : \mathcal{Y}_n \mapsto \mathbb{R}$ such that $\rho\left(Y^+_{kl}\right) - \rho\left(Y^-_{kl}\right) = u_a\left(Y^+_{kl}\right) - u_a\left(Y^-_{kl}\right)$ for all $a \in c_{kl}$ and all (k, l)
- In the above case, chance of a selecting $P^i_{jkl} = 1$ then becomes

$$\begin{aligned}
\Pr\left(P^{(i)}_{jkl} = \left(p^{(i-1)}\right)^+_{jkl} \middle| \left(P^{(i-1)}\right)^c_{jkl} = \left(p^{(i-1)}\right)^c_{jkl}, \rho\right) \\
= \operatorname{logit}^{-1}\left[\rho\left(r\left(\left(p^{(i-1)}\right)^+_{jkl}\right)\right) - \rho\left(r\left(\left(p^{(i-1)}\right)^-_{jkl}\right)\right)\right] \qquad (2)
\end{aligned}$$

  - So, where ρ exists, decision probabilities can be derived from effect on ρ (which is not agent-specific)
  - Many realistic models fall into this class (example will follow)

When Are Decisions Made?

- Some observations about the decision making process
  - Agents cognitively bounded – can't evaluate all ties simultaneously (or continuously)
  - Updating occurs in continuous time; exact simultaneity across agents a rare event
- Modeling framework: continuous time edge updating process
  - Unobserved, continuous time process gives agents opportunities to modify P
  - Formally, defined as process $X^{(1)}, X^{(2)}, \ldots$ of random (j, k, l, t) tuples
    - $a(X^{(i)}) = j$ is updating agent, $e_s(X^{(i)} = k$ and $e_r(X^{(i)}) = l$ are the sender/receiver of the hypothetical edge, and $\tau(X^{(i)}) = t$ is the event time
    - Assume X independent of P, and $\sum_{x:\tau(x)<t} I\left(a\left(x\right) = i, e_s(x) = j, e_r(x) = k\right) \to \infty$ as $t \to \infty$ a.s. for all $\{j, k\}$ (directed case (j, k)) in $E^*(\mathcal{Y}_n)$ and all $i \in c_{jk}$ (i.e., all edges, agents update at least occasionally)

Putting It All Together: Behavioral Equilibrium

- With the above, we demonstrate the following theorem:

  Theorem 1. *Let Y be the adjacency structure arising from the behavioral model specified by $(\mathcal{Y}_n, A, \ell, c, r, u)$ under edge updating process X, and let $Y^{[t]}$ be the state of Y at time t. If ρ is a potential for $(A, \ell, c, \mathcal{Y}_n)$, and X is such that*

  1. *X is independent of P; and*
  2. *$\sum_{x:\tau(x)<t} I\left(a\left(x\right) = i, e_s(x) = j, e_r(x) = k\right) \to \infty$ as $t \to \infty$ a.s. for all $\{j,k\}$ (directed case (j,k)) in $E^*(\mathcal{Y}_n)$ and all $i \in c_{jk}$,*

  then $Y^{[t]}$ converges in distribution to
  $\Pr\left(Y^{[t]} = y\right) = \left|\{p : r(p) = y\}\right| \frac{\exp[\rho(y)]}{\sum_{p' \in \mathcal{P}_n} \exp[\rho(r(p'))]}$ *on support $\mathcal{Y}_n$ as $t \to \infty$.*

- In other words, we can go from utilities (via ρ) to a well-specified equilibrium distribution!

Interpreting the Equilibrium

- Note that we can re-write equilibrium distribution in terms of Y:

$$\Pr\left(Y^{[t]} = y\right) = \frac{\left|\{p : r(p) = y\}\right| \exp\left[\rho\left(y\right)\right]}{\sum_{y' \in \mathcal{Y}_n} \left|\{p : r(p) = y'\}\right| \exp\left[\rho\left(y'\right)\right]} \quad (3)$$

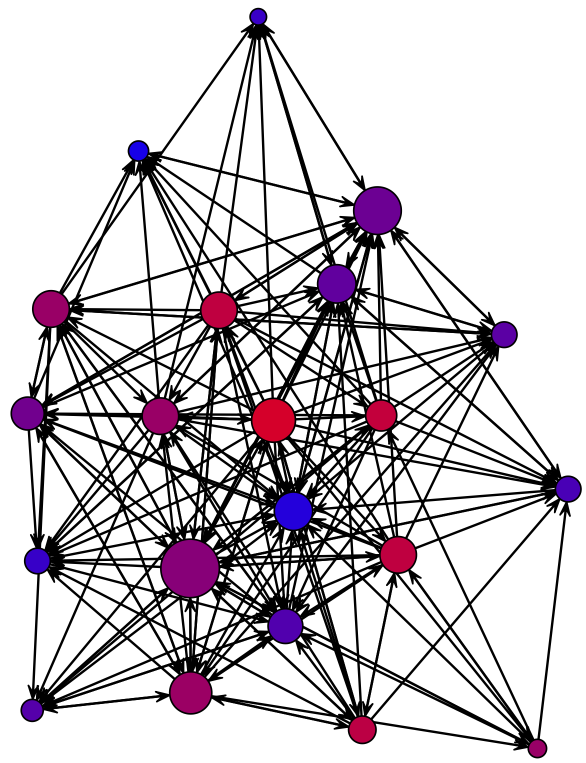
- This is an exponential random graph (ERG) form for Y, with graph potential $\ln\left|\{p : r(p) = y\}\right| + \rho(y)$
  - Preferred form for simulation/inference, with reasonably well-developed theory and tools (e.g., Handcock et al. (2003))
  - Behavior controlled by actor preferences, plus an offset due to the resolution function – both "rules" and preferences matter!

The Effect of Multilateral Control

- How, exactly, do common situations like multilateral edge control affect equilibrium?
- Let $s(Y) = |\{p : r(p) = y\}|$. Note that, when r is edgewise decomposable, $s(Y) = \prod s'(Y_{ij})$; if also homogeneous, becomes $s'(1)^{\sum Y_{ij}} s'(0)^{\sum(1-Y_{ij})}$
- Can show from the above that $\ln s(Y) = \left(\sum Y_{ij}\right) \ln\left(s'(1)/s'(0)\right) + \alpha$, where $s'(1)$ is the number of $P_{\cdot ij}$ combinations leading to $Y_{ij} = 1$, $s'(0)$ is the number of $P_{\cdot ij}$ combinations leading to $Y_{ij} = 0$, and α is a constant (can be dropped)
  - Thus, imposing multilateral control is equivalent to translating the edge term by a fixed amount that depends only on r!
  - In bilateral case, $s'(1)/s'(0)$ equals either 3 (epibolic) or 1/3 (symphonic); offset thus equals ± 1.1
- Important (good) news: to estimate ρ from observed Y, we can fit a standard ERG model to Y, and then adjust the estimated parameters for r
  - Under unilateral edge control, no correction is needed; more complex multilateral rules may require additional terms, but principle is same

Empirical Example: Advice-Seeking Among Managers

- Sample empirical application from Krackhardt (1987): self-reported advice-seeking among 21 managers in a high-tech firm
  - Additional covariates: friendship, authority (reporting)
- Demonstration: selection of potential behavioral mechanisms via ERGs
  - Models parameterized using utility components
  - Model parameters estimated using maximum likelihood (Geyer-Thompson)
  - Model selection via AIC



Advice-Seeking ERG – Model Comparison

- First cut: models with independent dyads:

	Deviance	Model df	AIC	Rank
Edges	578.43	1	580.43	7
Edges+Sender	441.12	21	483.12	4
Edges+Covar	548.15	3	554.15	5
Edges+Recip	577.79	2	581.79	8
Edges+Sender+Covar	385.88	23	431.88	2
Edges+Sender+Recip	405.38	22	449.38	3
Edges+Covar+Recip	547.82	4	555.82	6
Edges+Sender+Covar+Recip	**378.95**	**24**	**426.95**	**1**

- Elaboration: models with triadic dependence:

	Deviance	Model df	AIC	Rank
Edges+Sender+Covar+Recip	378.95	24	426.95	4
Edges+Sender+Covar+Recip+CycTriple	361.61	25	411.61	2
Edges+Sender+Covar+Recip+TransTriple	368.81	25	418.81	3
Edges+Sender+Covar+Recip+CycTriple+TransTriple	**358.73**	**26**	**410.73**	**1**

- Verdict: data supplies evidence for heterogeneous edge formation preferences (w/covariates), with additional effects for reciprocated, cycle-completing, and transitive-completing edges.

Advice-Seeking ERG – AIC Selected Model

Effect	$\hat{\theta}$	s.e.	$\Pr(>\|Z\|)$		Effect	$\hat{\theta}$	s.e.	$\Pr(>\|Z\|)$	
Edges	**−1.022**	**0.137**	**0.0000**	$***$	**Sender14**	**−1.513**	**0.231**	**0.0000**	$***$
Sender2	**−2.039**	**0.637**	**0.0014**	$**$	**Sender15**	**16.605**	**0.336**	**0.0000**	$***$
Sender3	0.690	0.466	0.1382		**Sender16**	**−1.472**	**0.232**	**0.0000**	$***$
Sender4	−0.049	0.441	0.9112		**Sender17**	**−2.548**	**0.197**	**0.0000**	$***$
Sender5	0.355	0.495	0.4734		**Sender18**	**1.383**	**0.214**	**0.0000**	$***$
Sender6	**−4.654**	**1.540**	**0.0025**	$**$	**Sender19**	**−0.601**	**0.190**	**0.0016**	$**$
Sender7	−0.108	0.375	0.7726		Sender20	0.136	0.161	0.3986	
Sender8	−0.449	0.479	0.3486		Sender21	0.105	0.210	0.6157	
Sender9	0.393	0.496	0.4281		**Reciprocity**	**0.885**	**0.081**	**0.0000**	$***$
Sender10	0.023	0.555	0.9662		**Edgecov (Reporting)**	**5.178**	**0.947**	**0.0000**	$***$
Sender11	**−2.864**	**0.721**	**0.0001**	$***$	**Edgecov (Friendship)**	**1.642**	**0.132**	**0.0000**	$***$
Sender12	**−2.736**	**0.331**	**0.0000**	$***$	**CycTriple**	**−0.216**	**0.013**	**0.0000**	$***$
Sender13	**−0.986**	**0.194**	**0.0000**	$***$	**TransTriple**	**0.090**	**0.003**	**0.0000**	$***$

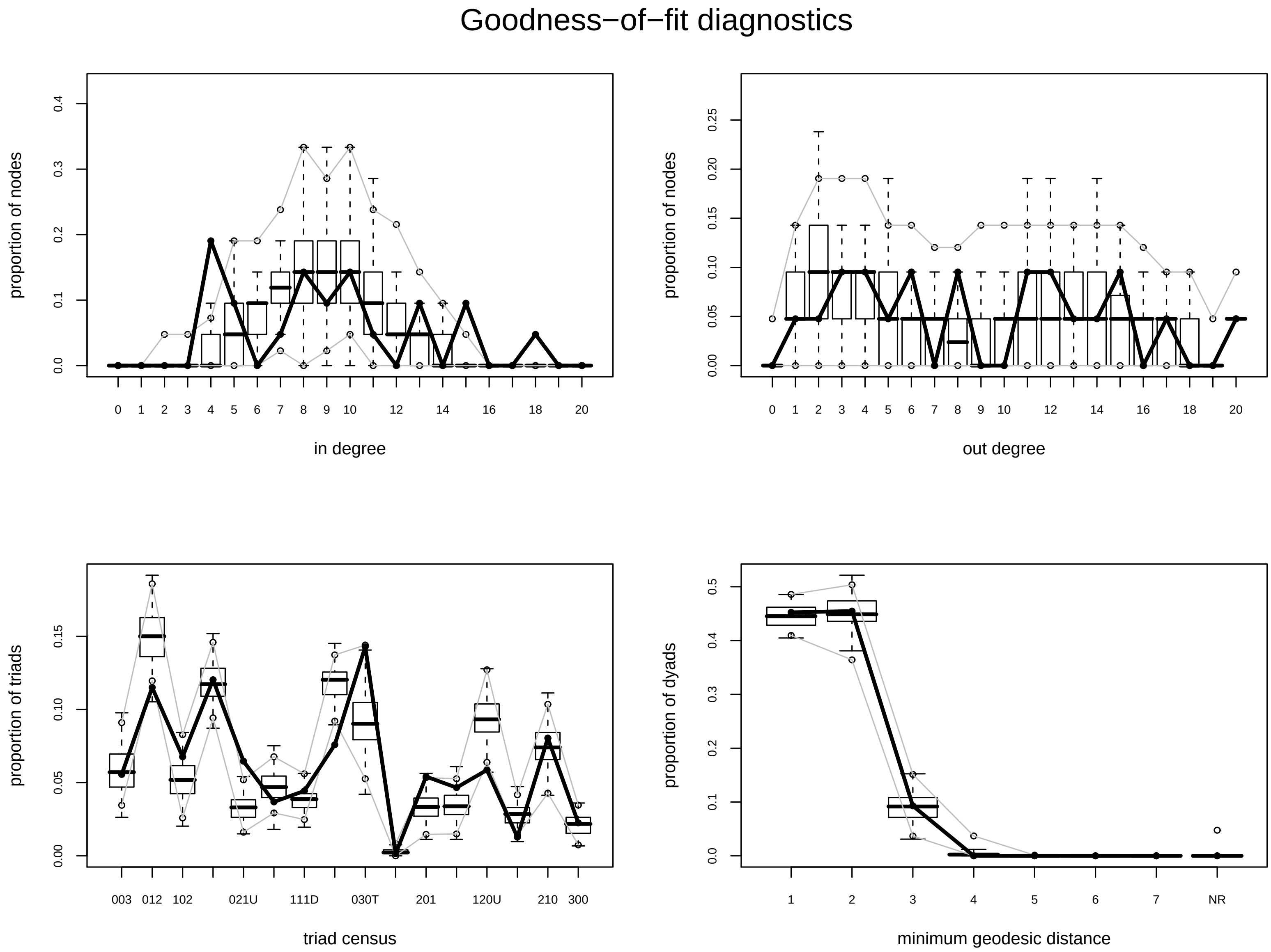
Null Dev 582.24; Res Dev 358.73 on 394 df

- Some observations...
  - Arbitrary edges are costly for most actors
  - Edges to friends and superiors are “cheaper” (or even positive payoff)
  - Reciprocating edges, edges with transitive completion are cheaper...
  - ...but edges which create (in)cycles are more expensive; a sign of hierarchy?

Summary

- Linking low-level processes and aggregate outcomes is a non-trivial problem
  - Not every process leads to intelligible results
  - Not all of the above are behaviorally plausible
- Potential games for cross-sectional (ERG) network models
  - Allow us to derive random cross-sectional behavior from strategic interaction
  - Provide sufficient conditions for ERG parameters to be interpreted in terms of preferences
  - Allows for testing of competing behavioral models (assuming scope conditions are met!)
- Approach seems promising, but many questions remain
  - Can we characterize utilities which lead to identifiable models?
  - How can we leverage other properties of potential games?

Model Adequacy Check



Exponential Families for Random Graphs

- For random graph G w/countable support $\mathcal{G}$, pmf is given in ERG form by

$$\Pr(G = g|\theta) = \frac{\exp\left(\theta^T \mathbf{t}(g)\right)}{\sum_{g' \in \mathcal{G}} \exp\left(\theta^T \mathbf{t}(g')\right)} I_{\mathcal{G}}(g) \tag{4}$$

- $\theta^T \mathbf{t}$: linear predictor
  - $\mathbf{t} : \mathcal{G} \to \mathbb{R}^m$: vector of sufficient statistics
  - $\theta \in \mathbb{R}^m$: vector of parameters
  - $\sum_{g' \in \mathcal{G}} \exp\left(\theta^T \mathbf{t}(g')\right)$: normalizing factor (aka partition function, Z)
- Intuition: ERG places more/less weight on structures with certain features, as determined by $\mathbf{t}$ and θ
  - Model is complete for pmfs on $\mathcal{G}$, few constraints on $\mathbf{t}$

Building Potentials: Independent Edge Effects

- General procedure
  - Identify utility for actor i
  - Determine difference in u_i for single edge change
  - Find ρ such that utility difference is equal to utility difference for all u_i
- Linear combinations of payoffs
  - If $u_i(\mathbf{y}) = \sum_j u_i^{(j)}(\mathbf{y})$, $\rho(\mathbf{y}) = \sum_j \rho_i^{(j)}(\mathbf{y})$
- Edge payoffs (homogeneous)
  - $u_i(\mathbf{y}) = \theta \sum_j y_{ij}$
  - $u_i\left(\mathbf{y}_{ij}^{+}\right) - u_i\left(\mathbf{y}_{ij}^{-}\right) = \theta$
  - $\rho(\mathbf{y}) = \theta \sum_i \sum_j y_{ij}$
  - Equivalence: p_1/Bernoulli density effect
- Edge payoffs (inhomogeneous)
  - $u_i(\mathbf{y}) = \theta_i \sum_j y_{ij}$
  - $u_i\left(\mathbf{y}_{ij}^{+}\right) - u_i\left(\mathbf{y}_{ij}^{-}\right) = \theta_i$
  - $\rho(\mathbf{y}) = \sum_i \theta_i \sum_j y_{ij}$
  - Equivalence: p_1 expansiveness effect
- Edge covariate payoffs
  - $u_i(\mathbf{y}) = \theta \sum_j y_{ij} x_{ij}$
  - $u_i\left(\mathbf{y}_{ij}^{+}\right) - u_i\left(\mathbf{y}_{ij}^{-}\right) = \theta x_{ij}$
  - $\rho(\mathbf{y}) = \theta \sum_i \sum_j y_{ij} x_{ij}$
  - Equivalence: Edgewise covariate effects (netlogit)

Building Potentials: Dependent Edge Effects

- Reciprocity payoffs
  - $u_i(\mathbf{y}) = \theta \sum_j y_{ij} y_{ji}$
  - $u_i\left(\mathbf{y}^+_{ij}\right) - u_i\left(\mathbf{y}^-_{ij}\right) = \theta y_{ji}$
  - $\rho(\mathbf{y}) = \theta \sum_i \sum_{j<i} y_{ij} y_{ji}$
  - Equivalence: p_1 reciprocity effect
- 3-Cycle payoffs
  - $u_i(\mathbf{y}) = \theta \sum_{j \neq i} \sum_{k \neq i,j} y_{ij} y_{jk} y_{ki}$
  - $u_i\left(\mathbf{y}^+_{ij}\right) - u_i\left(\mathbf{y}^-_{ij}\right) = \theta \sum_{k \neq i,j} y_{jk} y_{ki}$
  - $\rho(\mathbf{y}) = \frac{\theta}{3} \sum_i \sum_{j \neq i} \sum_{k \neq i,j} y_{ij} y_{jk} y_{ki}$
  - Equivalence: Cyclic triple effect
- Transitive completion payoffs
  - $u_i(\mathbf{y}) = \theta \sum_{j \neq i} \sum_{k \neq i,j} \begin{bmatrix} y_{ij} y_{ki} y_{kj} + y_{ij} y_{ik} y_{jk} \\ + y_{ij} y_{ik} y_{kj} \end{bmatrix}$
  - $u_i\left(\mathbf{y}^+_{ij}\right) - u_i\left(\mathbf{y}^-_{ij}\right) = \theta \sum_{k \neq i,j} \left[y_{ki} y_{kj} + y_{ik} y_{jk} + y_{ik} y_{kj} \right]$
  - $\rho(\mathbf{y}) = \theta \sum_i \sum_{j \neq i} \sum_{k \neq i,j} y_{ij} y_{ik} y_{kj}$
  - Equivalence: Transitive triple effect

Additional Insights from Potential Game Theory

- Game-theoretic properties of the behavioral model
  - Local maxima of ρ over $\mathcal{Y}_n$ correspond to Nash equilibria in pure strategies; global maxima of ρ correspond to stochastically stable Nash equilibria in pure strategies
    - At least one maximum must exist, since ρ is bounded above for any given θ
  - Fictitious play property; Nash equilibria compatible with best responses to mean strategy profile for population (interpreted as a mixed strategy)
- Implications for simulation, model behavior
  - Multiplying θ by a constant $\alpha \to \infty$ will drive the system to its SSNE
    - Likewise, best response dynamics (equivalent to conditional stepwise ascent) always leads to a NE
  - For degenerate models, “frozen” structures represent Nash equilibria in the associated potential game
    - Suggests a social interpretation of degeneracy in at least some cases: either correctly identifies robust social regimes, or points to incorrect preference structure

Proof Sketch for Potential Game Theorem (Unilateral Dyadic Case)

Assume an updating opportunity arises for y_{ij}, and assume that player k has control of y_{ij}. By the logistic choice assumption,

$$\Pr\left(\mathbf{Y} = \mathbf{y}^{+}_{ij} \middle| \mathbf{Y}^{c}_{ij} = \mathbf{y}^{c}_{ij}\right) = \frac{\exp\left(u_k\left(\mathbf{y}^{+}_{ij}\right)\right)}{\exp\left(u_k\left(\mathbf{y}^{+}_{ij}\right)\right) + \exp\left(u_k\left(\mathbf{y}^{-}_{ij}\right)\right)} \quad (5)$$

$$= \left[1 + \exp\left(u_k\left(\mathbf{y}^{-}_{ij}\right) - u_k\left(\mathbf{y}^{+}_{ij}\right)\right)\right]^{-1}. \quad (6)$$

Since $u, \mathcal{Y}$ form a potential game, $\exists\, \rho : \rho\left(\mathbf{y}^{+}_{ij}\right) - \rho\left(\mathbf{y}^{-}_{ij}\right) = u_k\left(\mathbf{y}^{+}_{ij}\right) - u_k\left(\mathbf{y}^{-}_{ij}\right) \;\forall\, k, (i,j), \mathbf{y}^{c}_{ij}$. Therefore, $\Pr\left(\mathbf{Y} = \mathbf{y}^{+}_{ij} \middle| \mathbf{Y}^{c}_{ij} = \mathbf{y}^{c}_{ij}\right) = \left[1 + \exp\left(\rho\left(\mathbf{y}^{-}_{ij}\right) - \rho\left(\mathbf{y}^{+}_{ij}\right)\right)\right]^{-1}$. Now assume that the updating opportunities for $\mathbf{Y}$ occur sequentially such that (i,j) is selected independently of $\mathbf{Y}$, with positive probability for all (i,j). Given arbitrary starting point $\mathbf{Y}^{(0)}$, denote the updated sequence of matrices by $\mathbf{Y}^{(0)}, \mathbf{Y}^{(1)}, \ldots$. This sequence clearly forms an irreducible and aperiodic Markov chain on $\mathcal{Y}$ (so long as ρ is finite); it is known that this chain is a random scan Gibbs sampler on $\mathcal{Y}$ with equilibrium distribution $\Pr(\mathbf{Y} = \mathbf{y}) = \frac{\exp(\rho(\mathbf{y}))}{\sum_{\mathbf{y}' \in \mathcal{Y}} \exp(\rho(\mathbf{y}'))}$, which is an ERG with potential ρ. By the ergodic theorem, then $\mathbf{Y}^{(i)} \xrightarrow[i \to \infty]{} ERG(\rho(\mathbf{Y}))$. QED.

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