MEMBERSHIP DIMENSION

CATEGORY-BASED ROUTING IN SOCIAL NETWORKS
AND THE
SMALL-WORLD PHENOMENON

David Eppstein ◆ Michael Goodrich ◆ Maarten Löffler

Darren Strash ◆ Lowell Trott

PART I
INTRODUCTION

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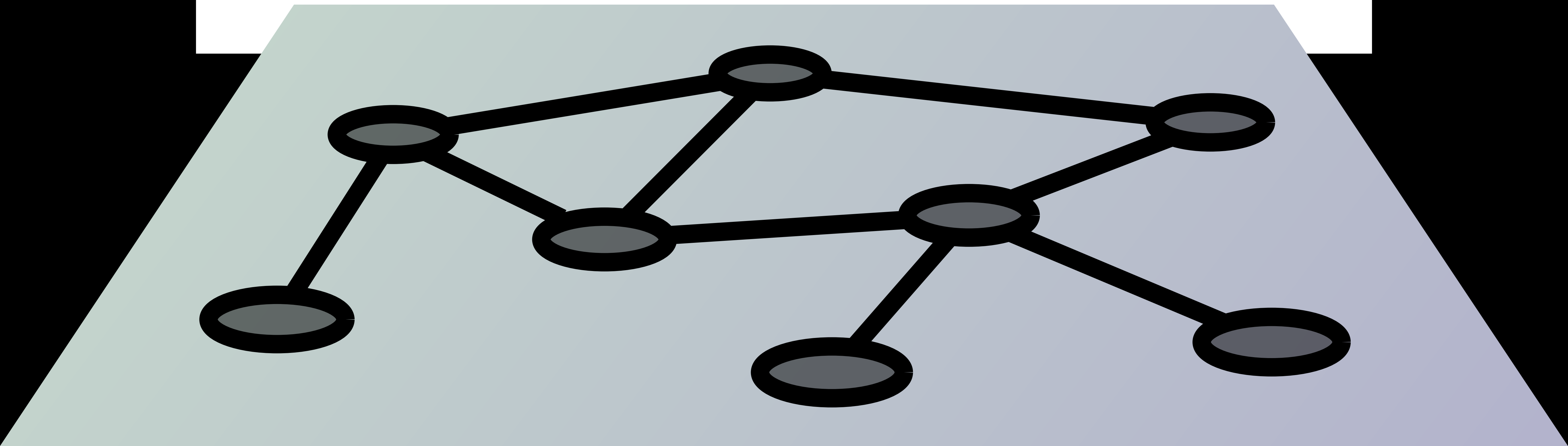

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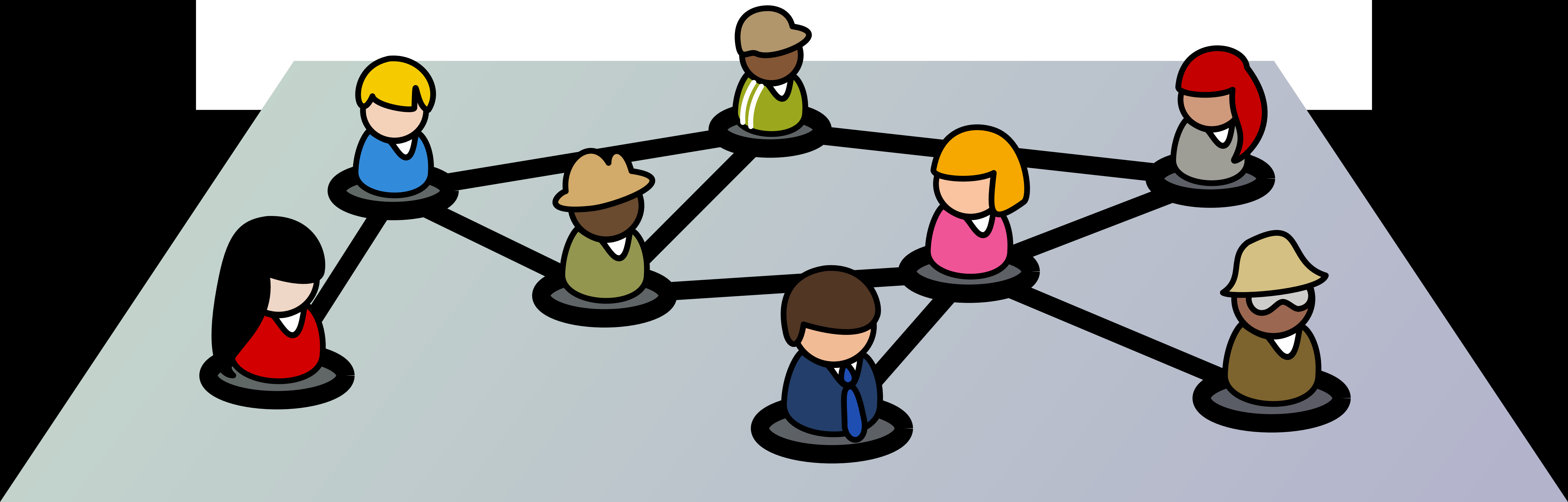

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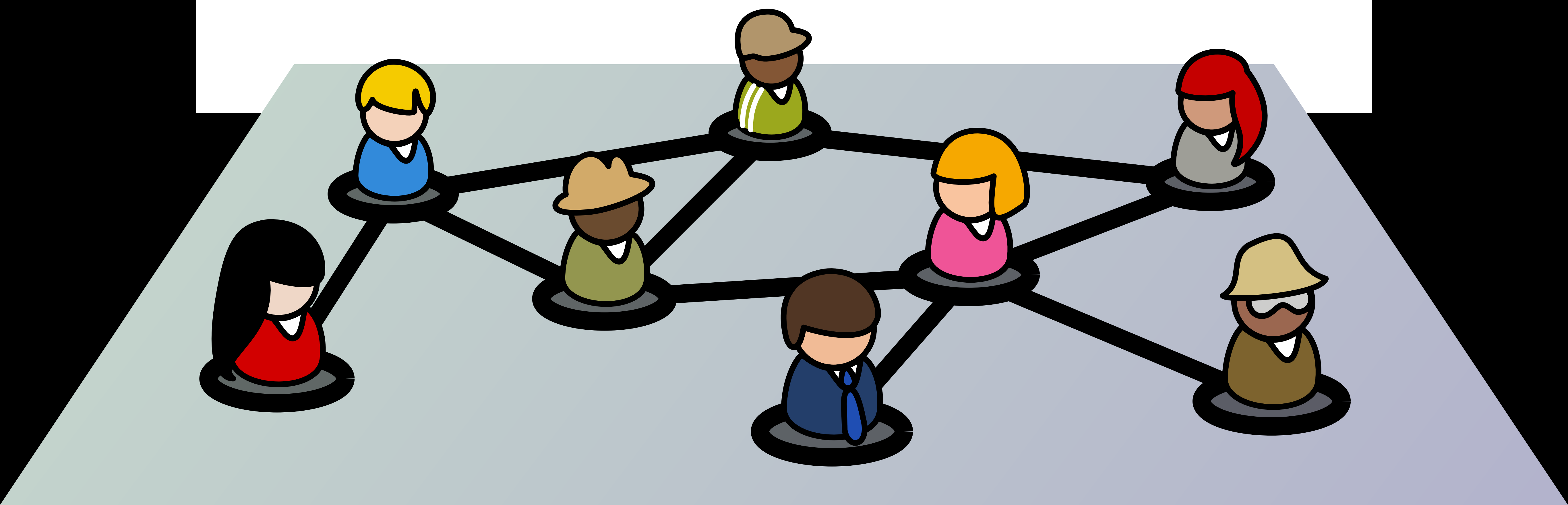

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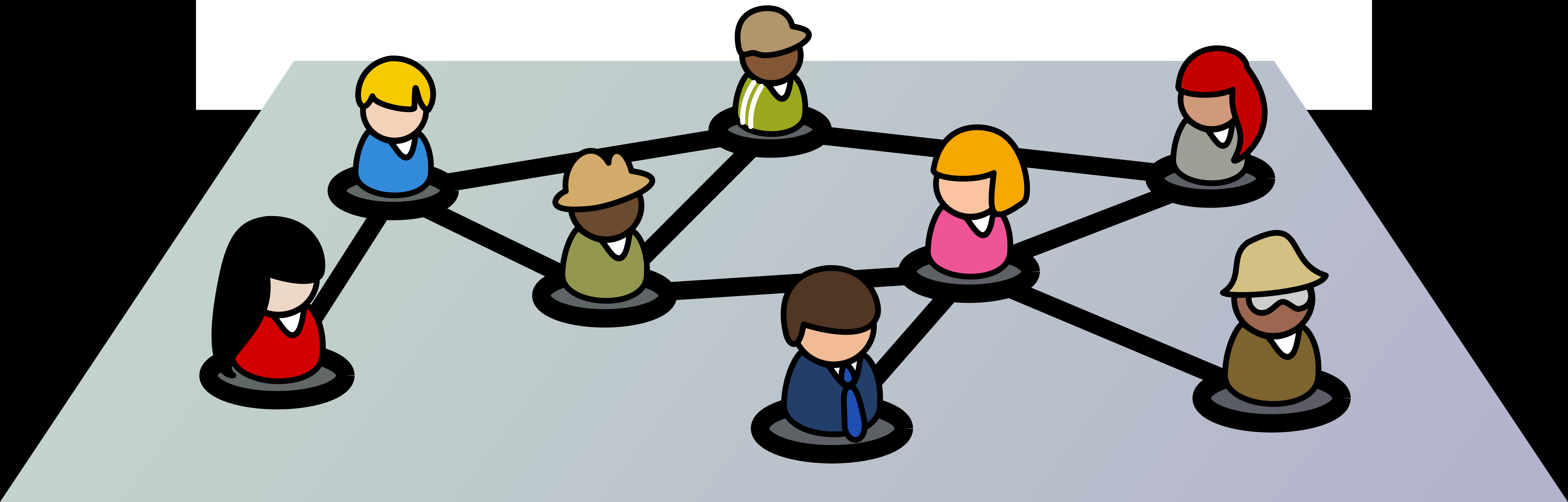

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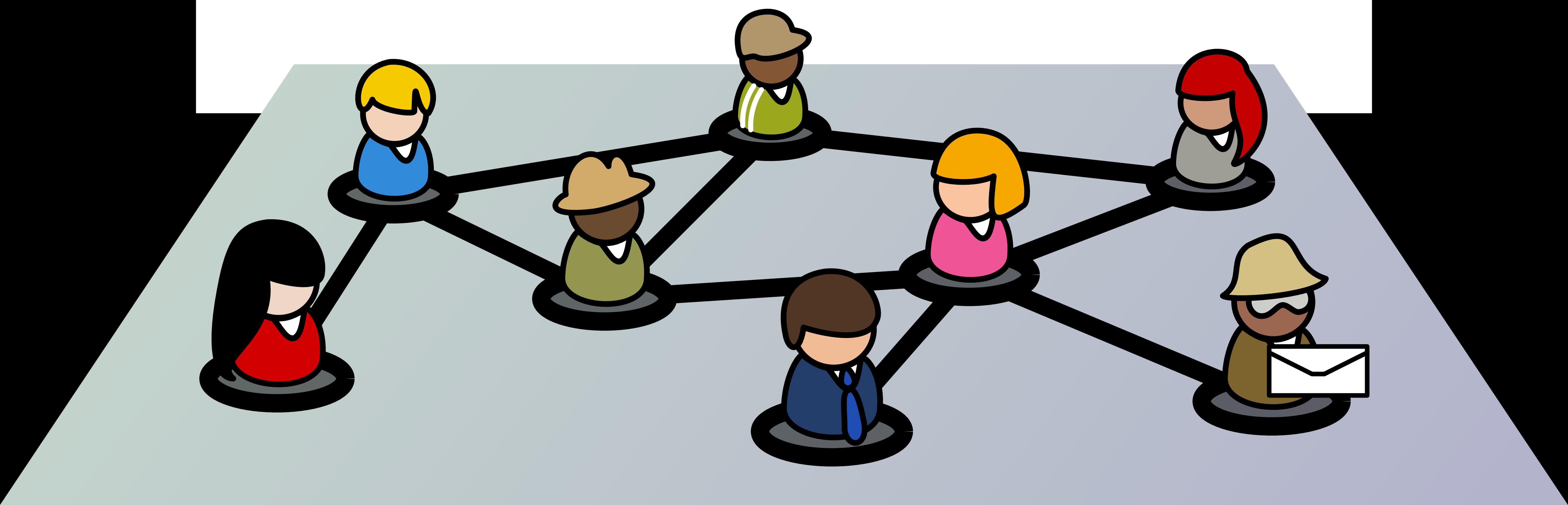

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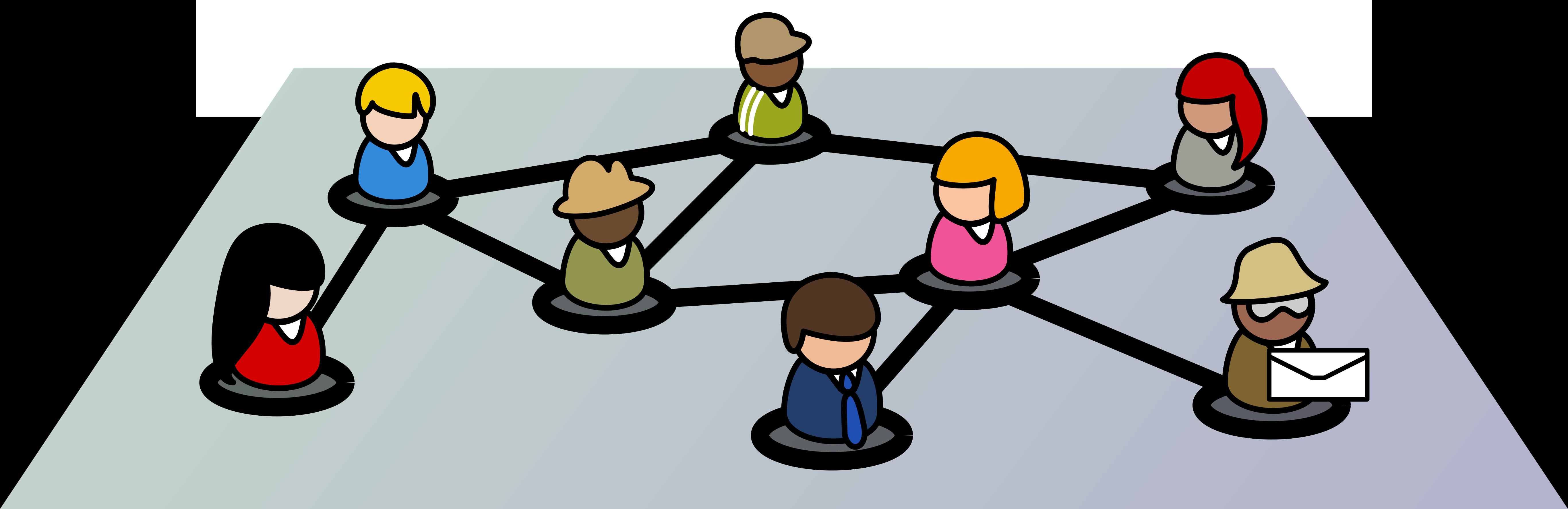

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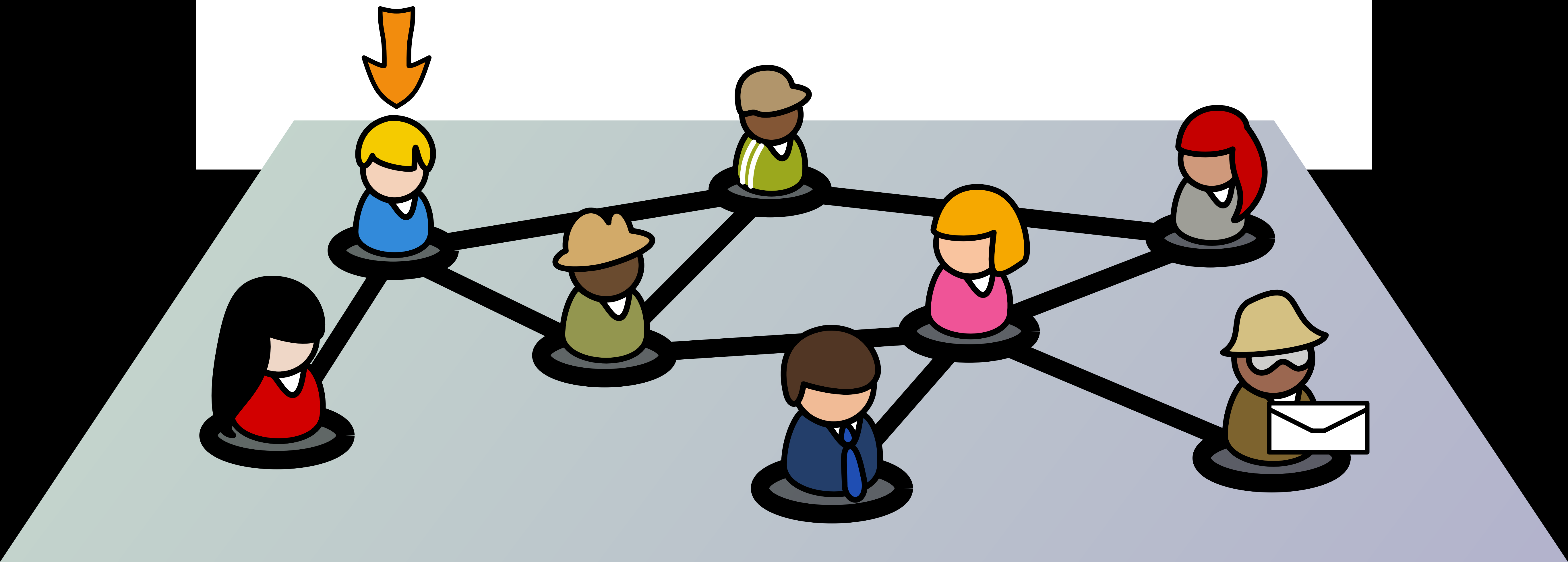

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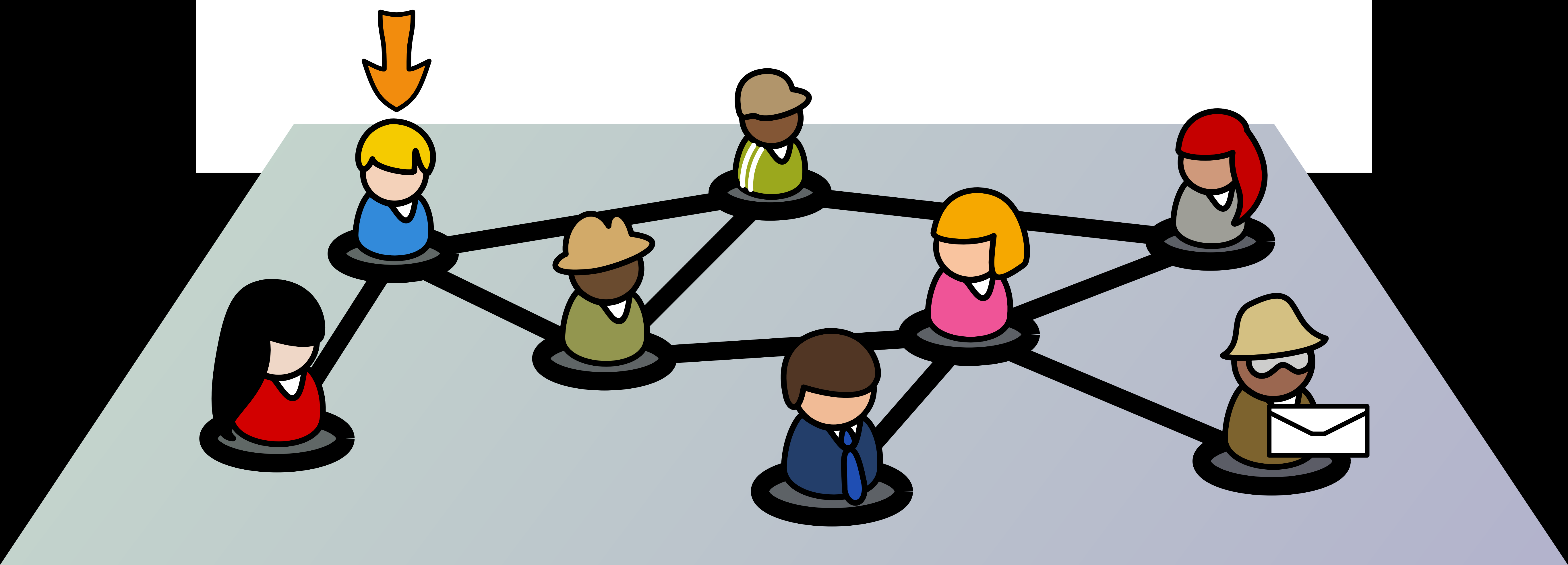

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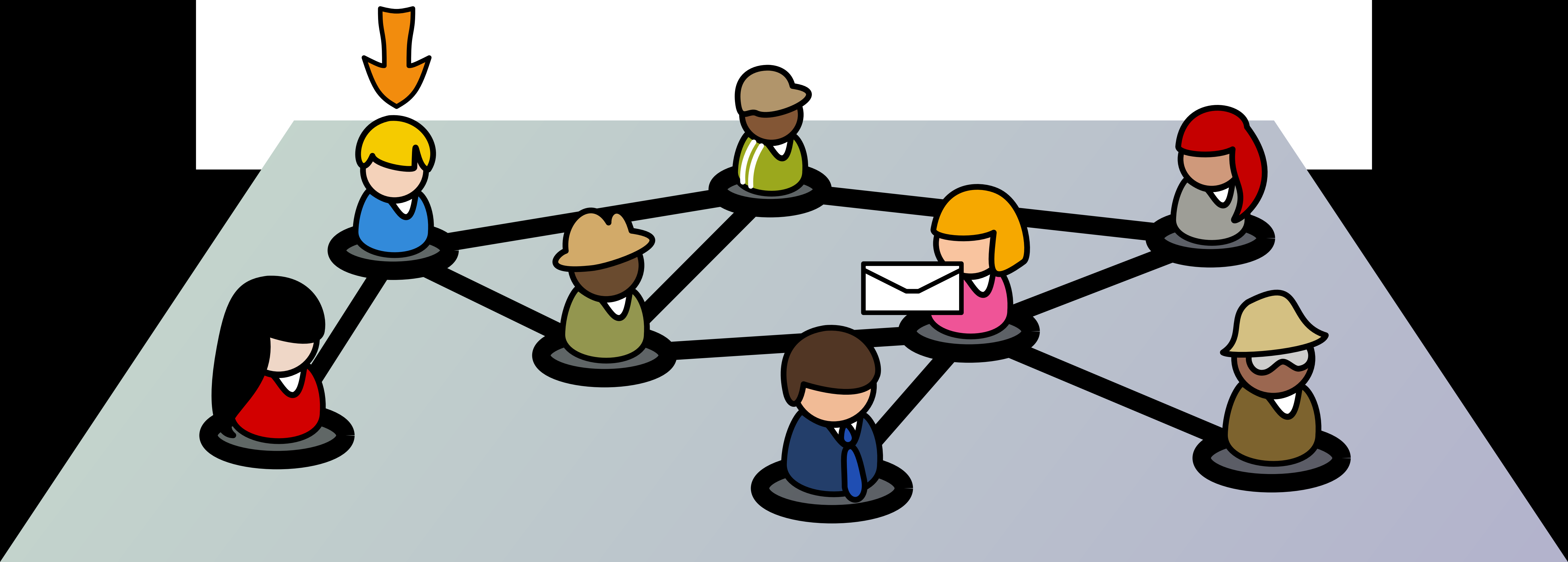

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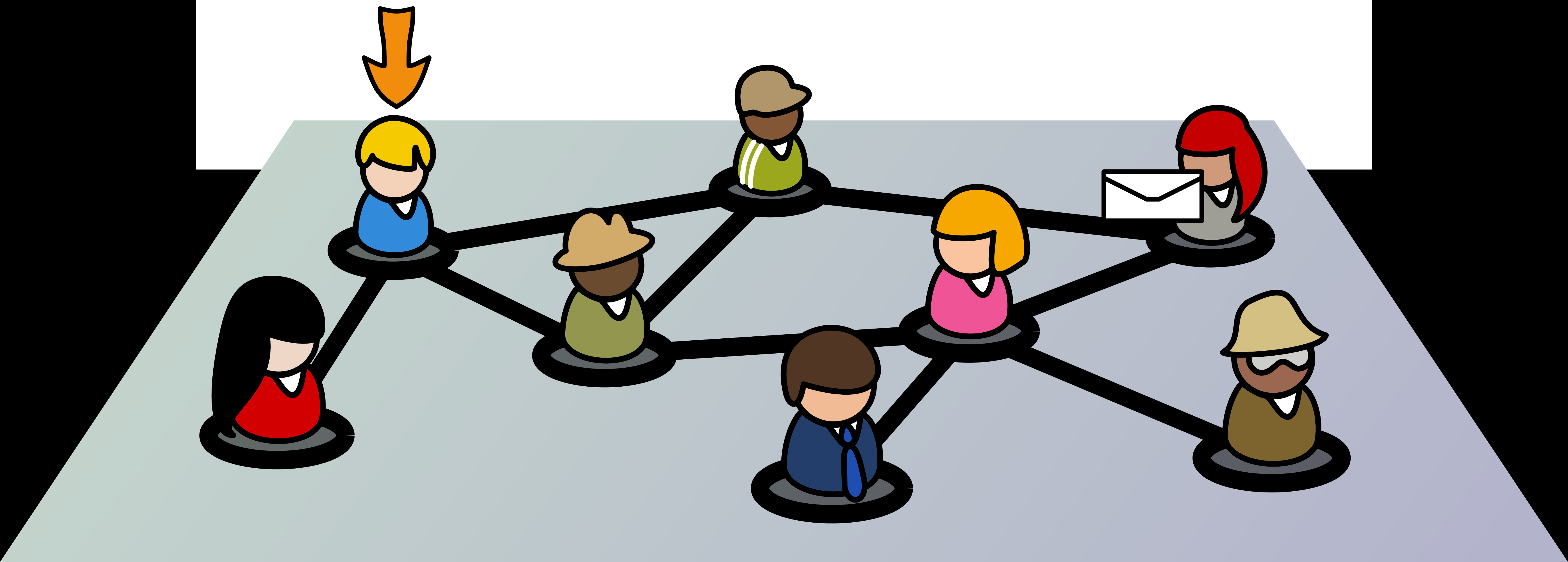

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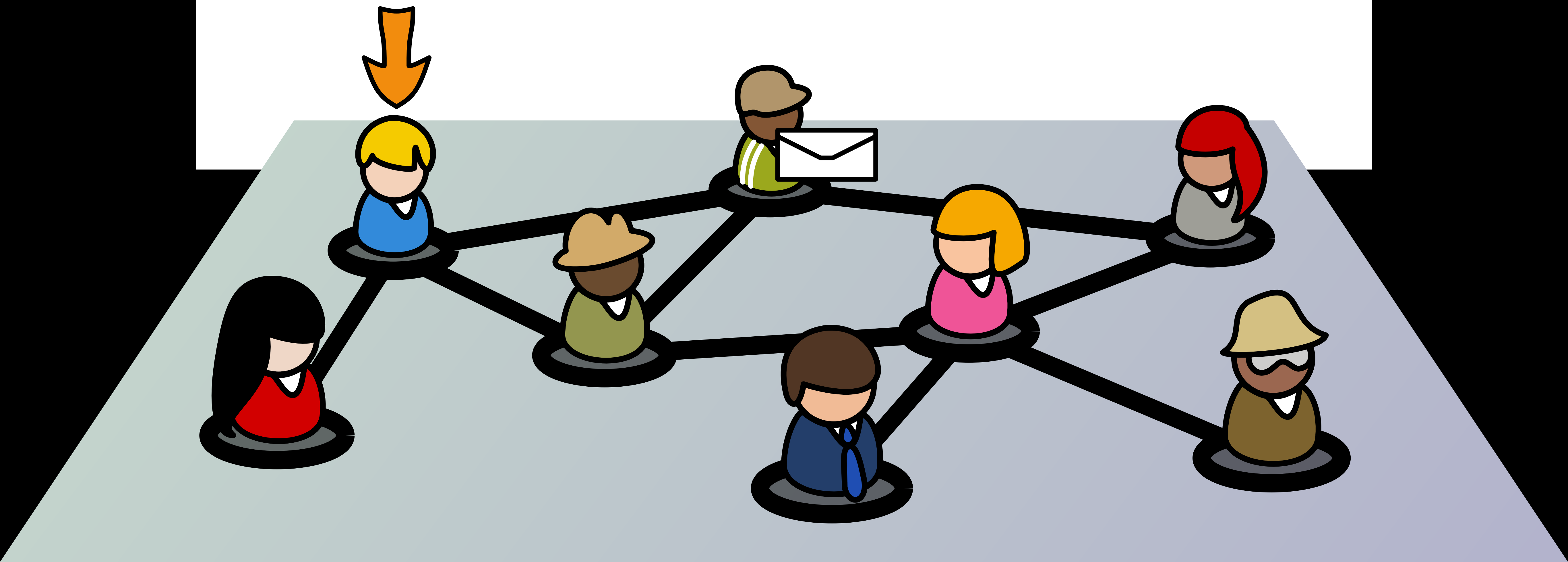

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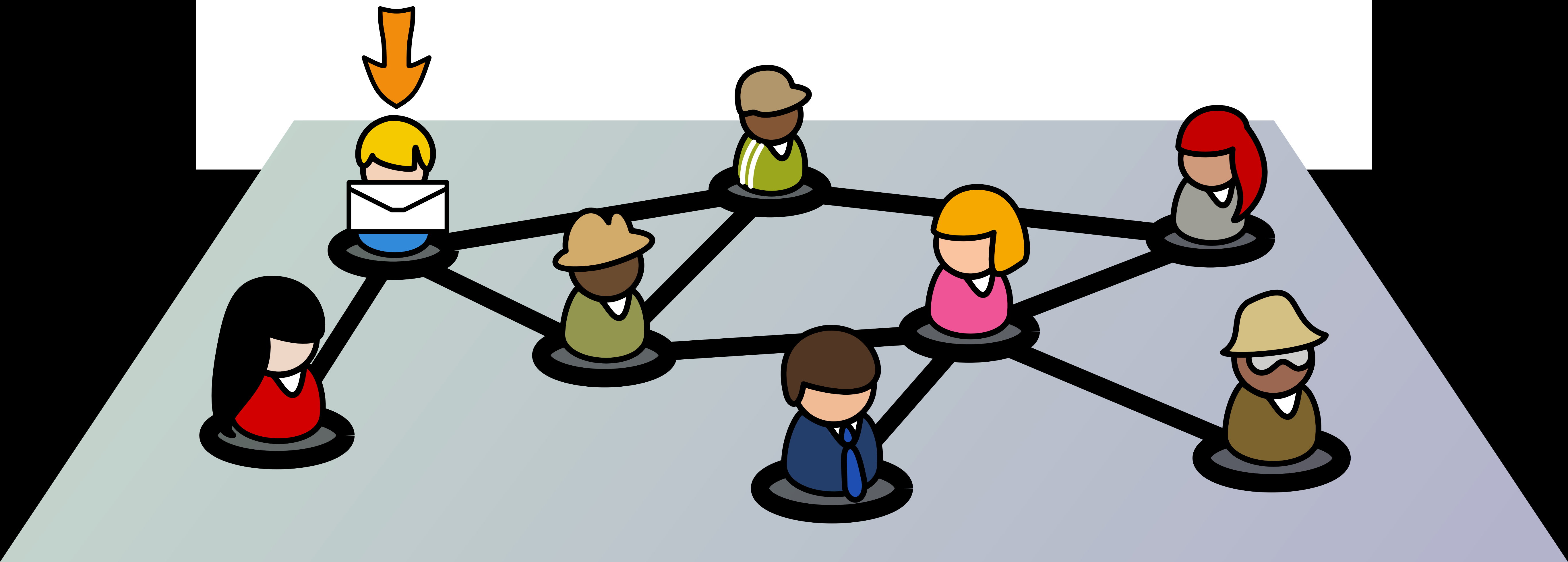

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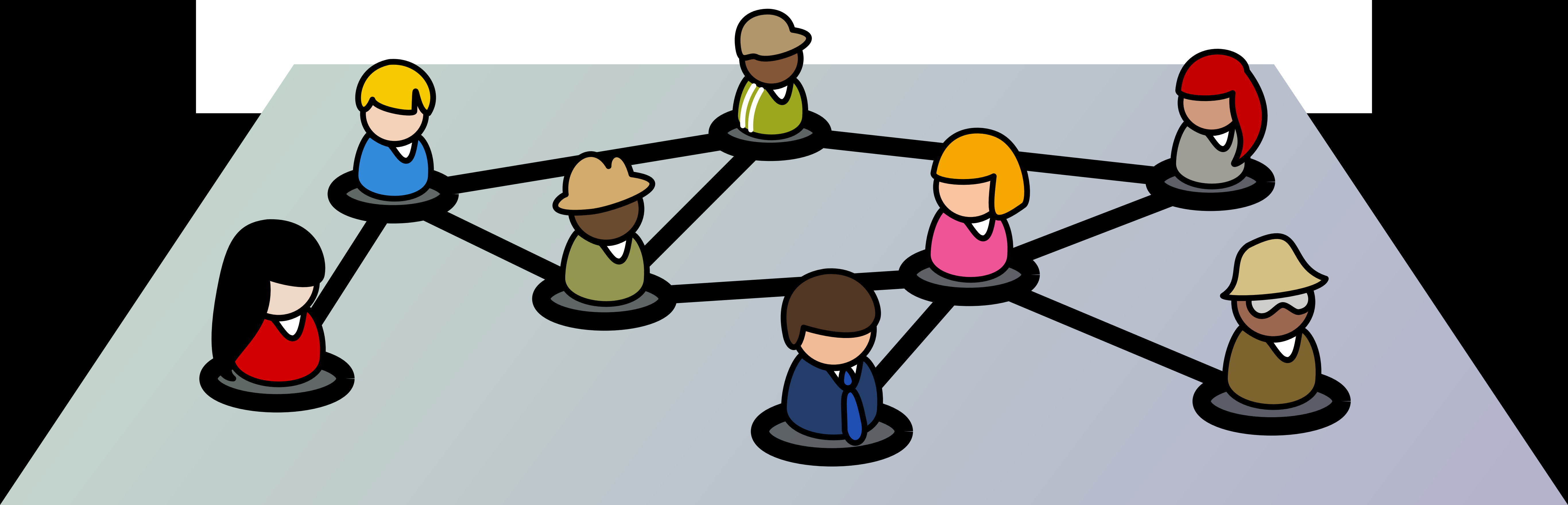

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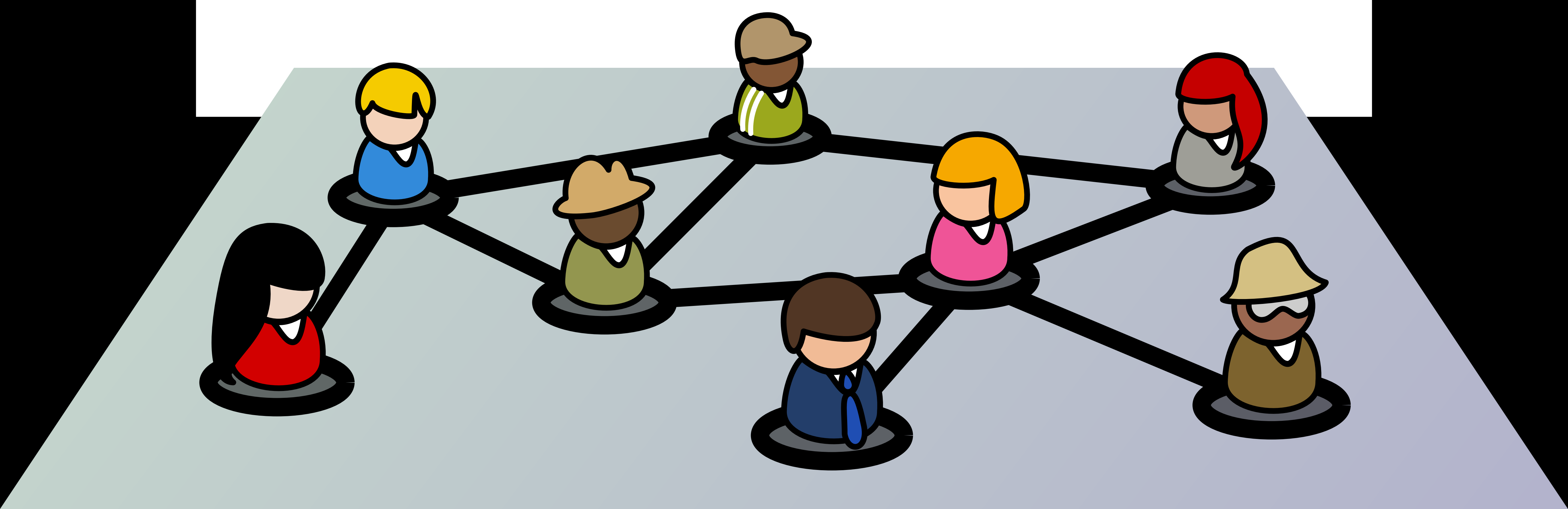

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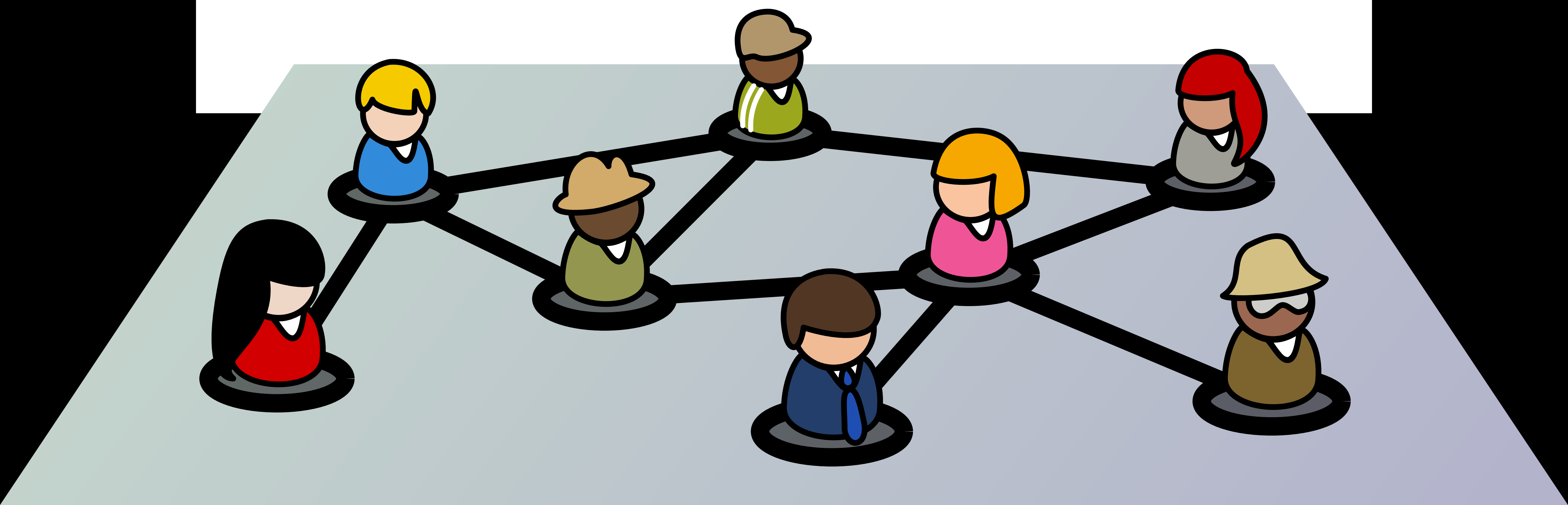

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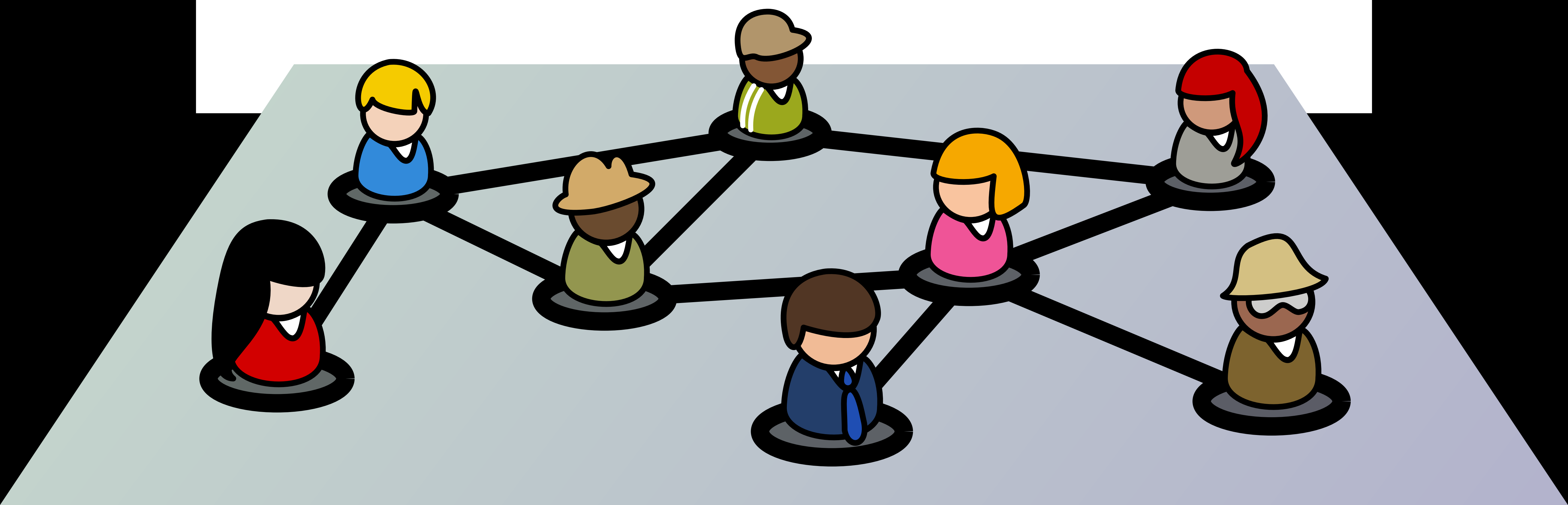

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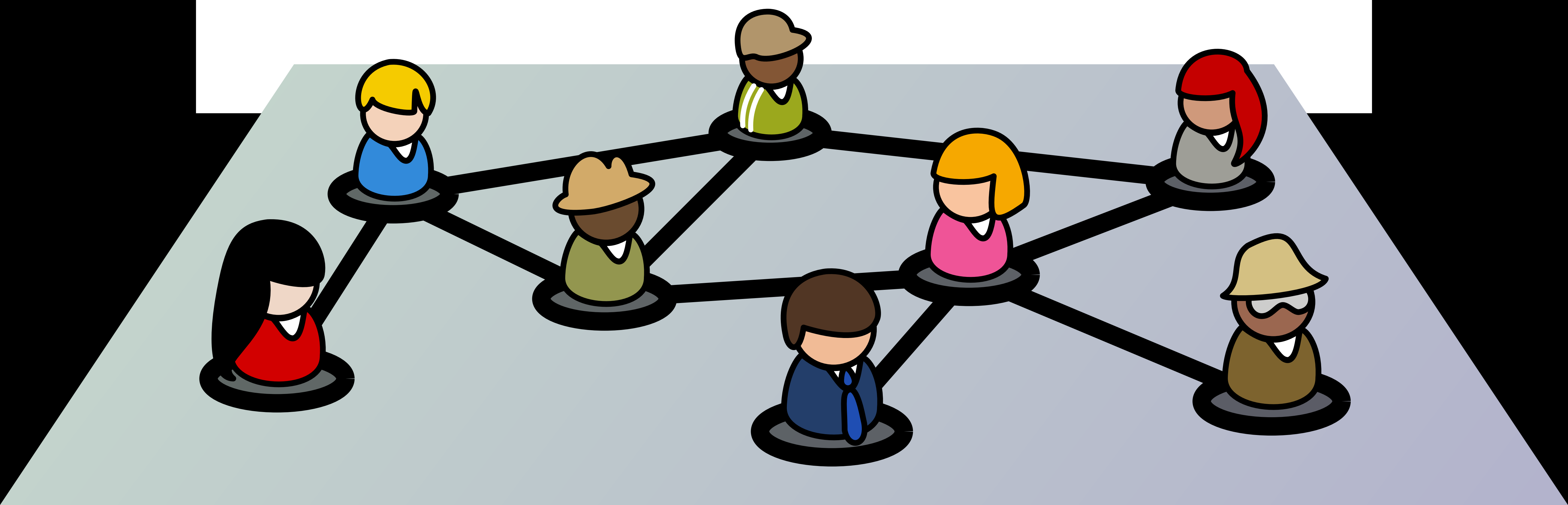

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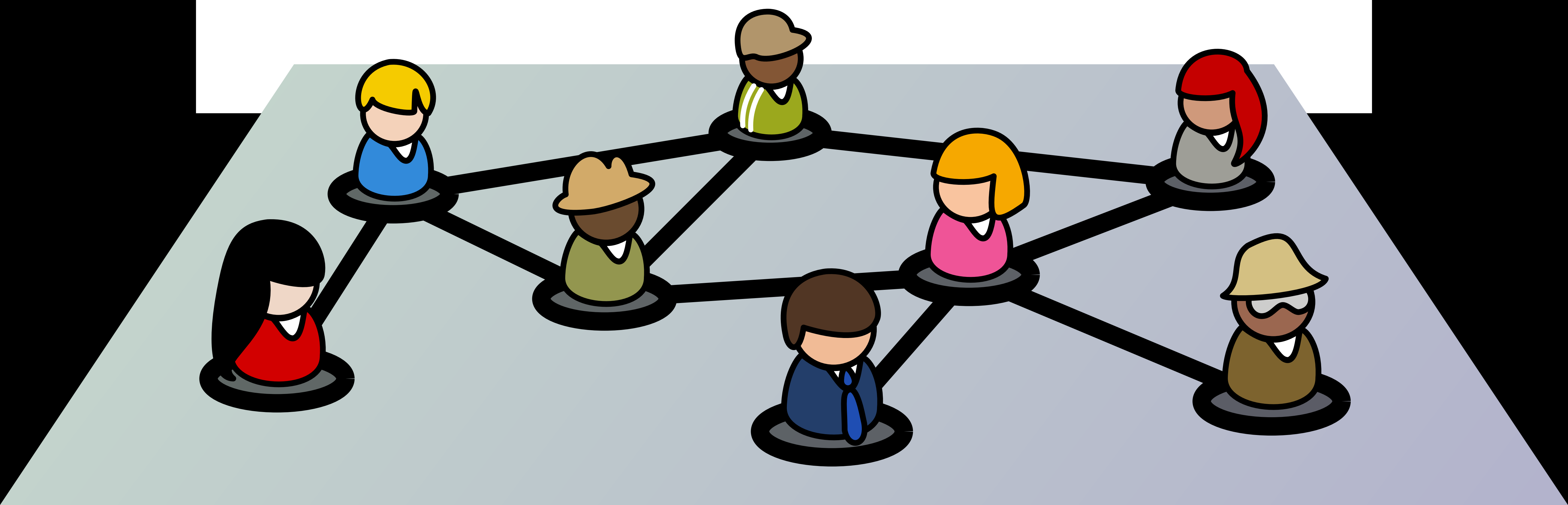

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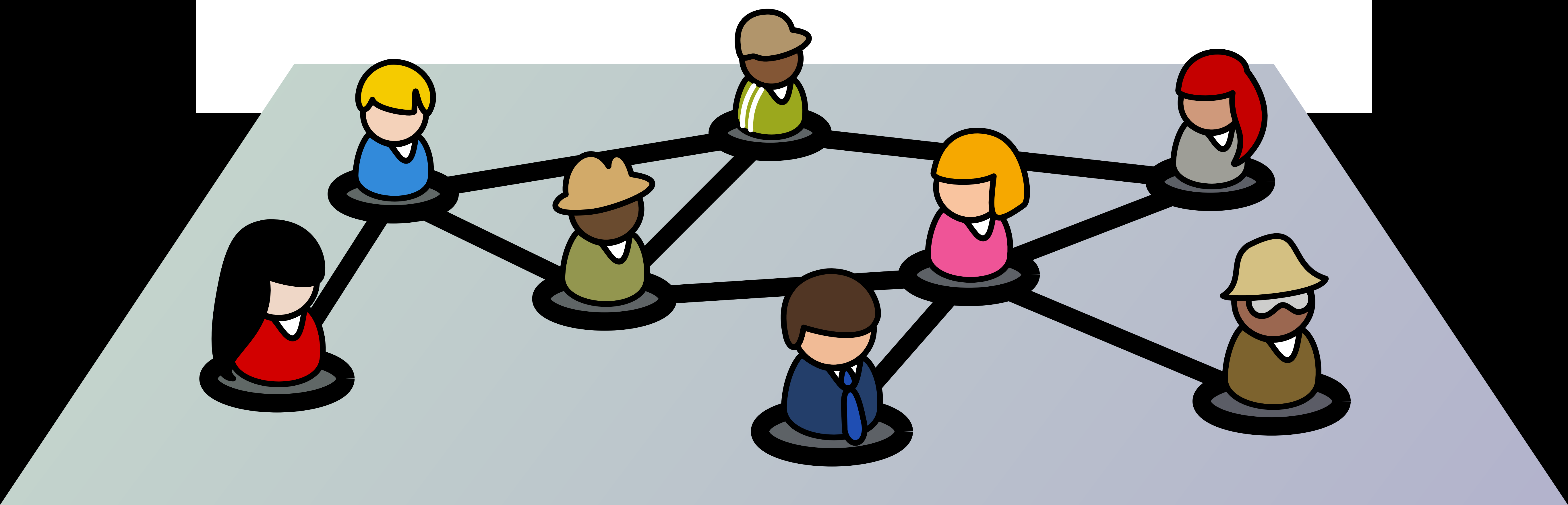

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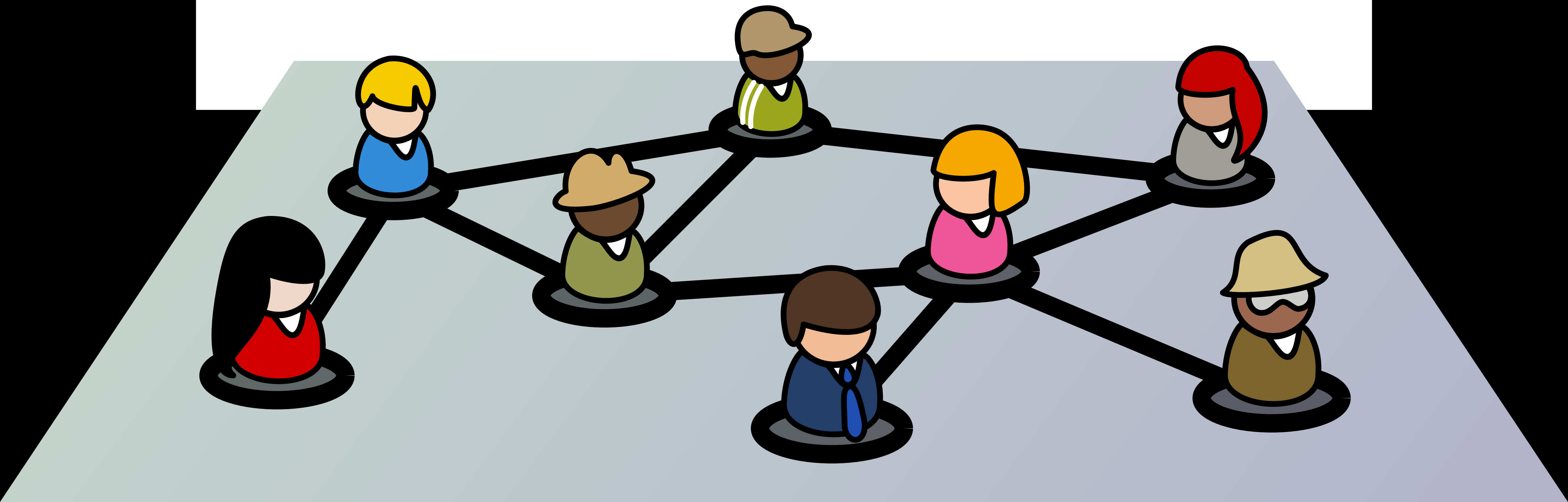

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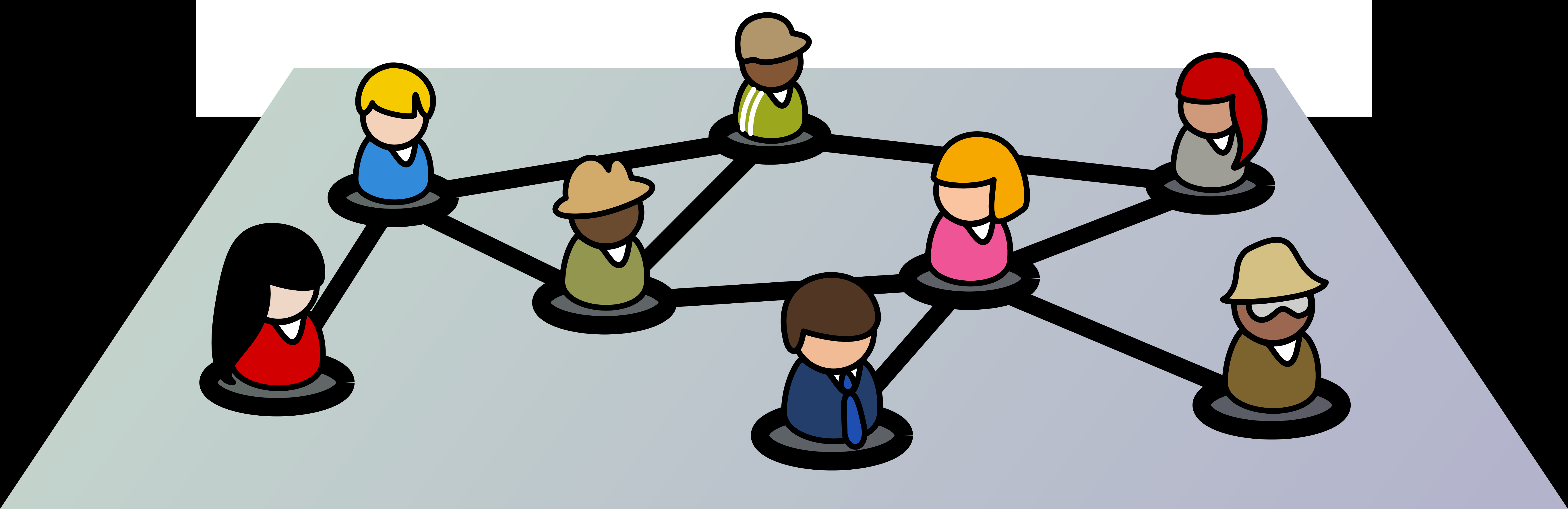

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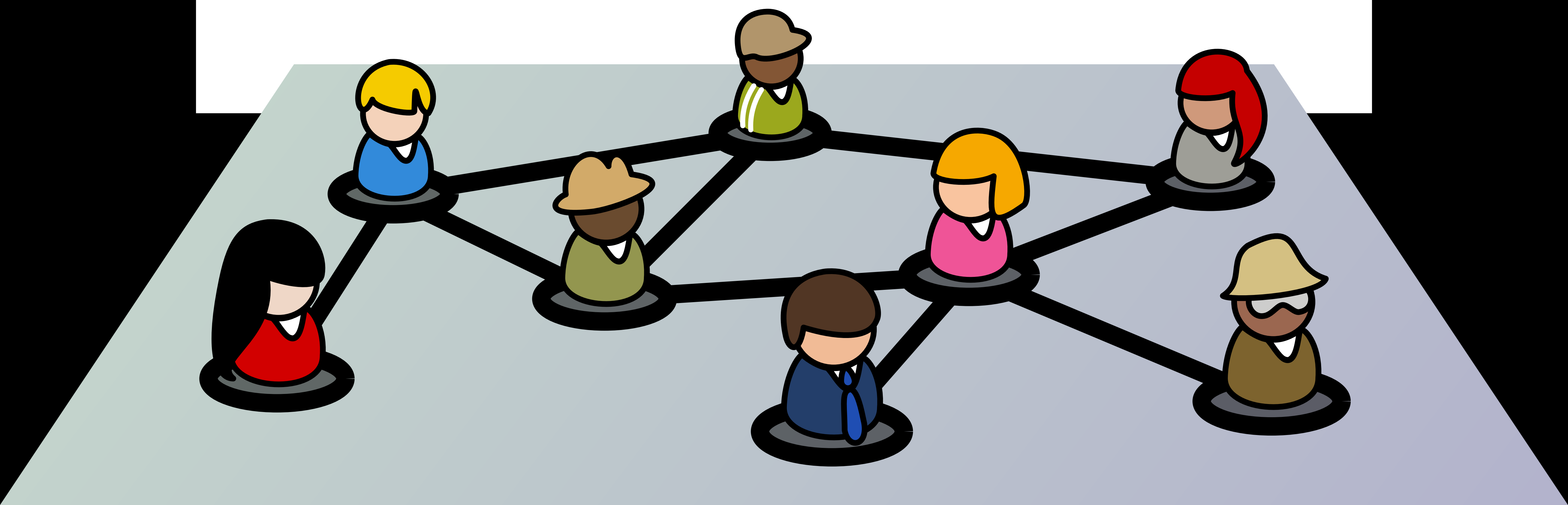

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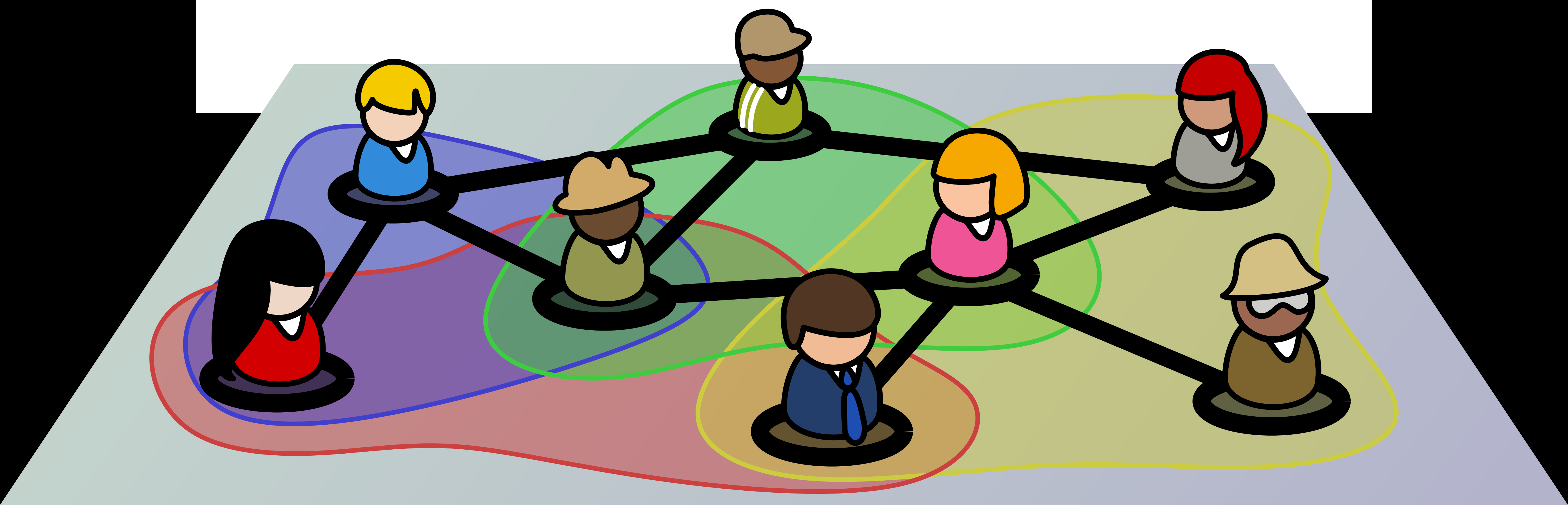

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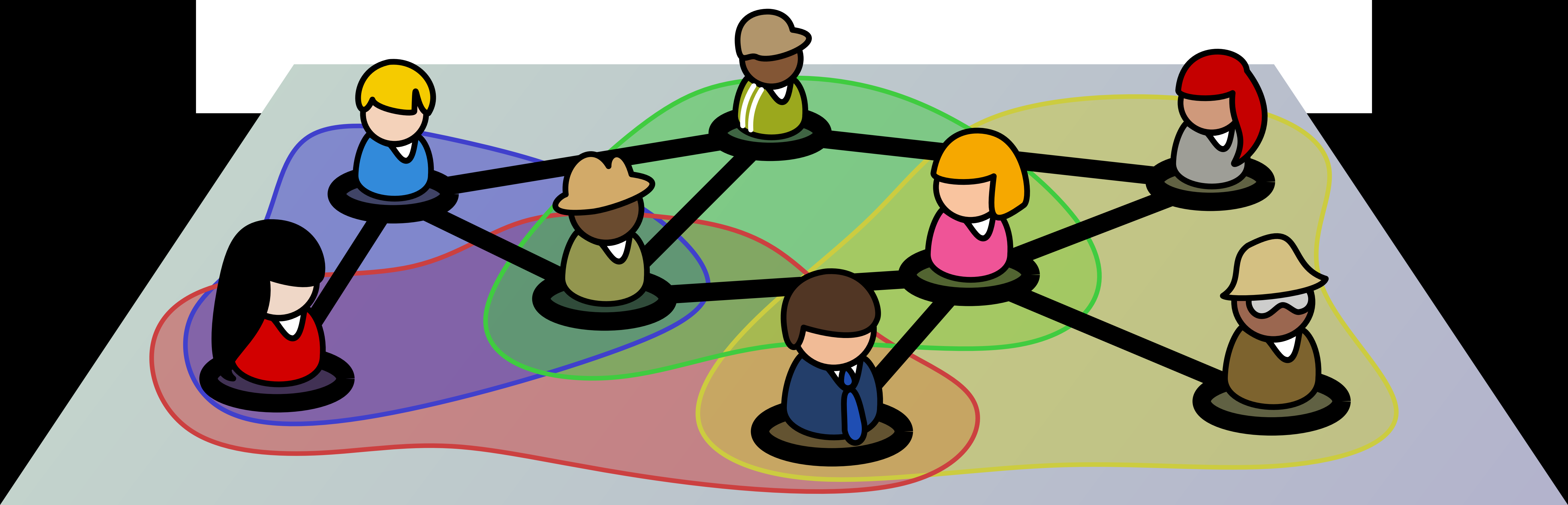

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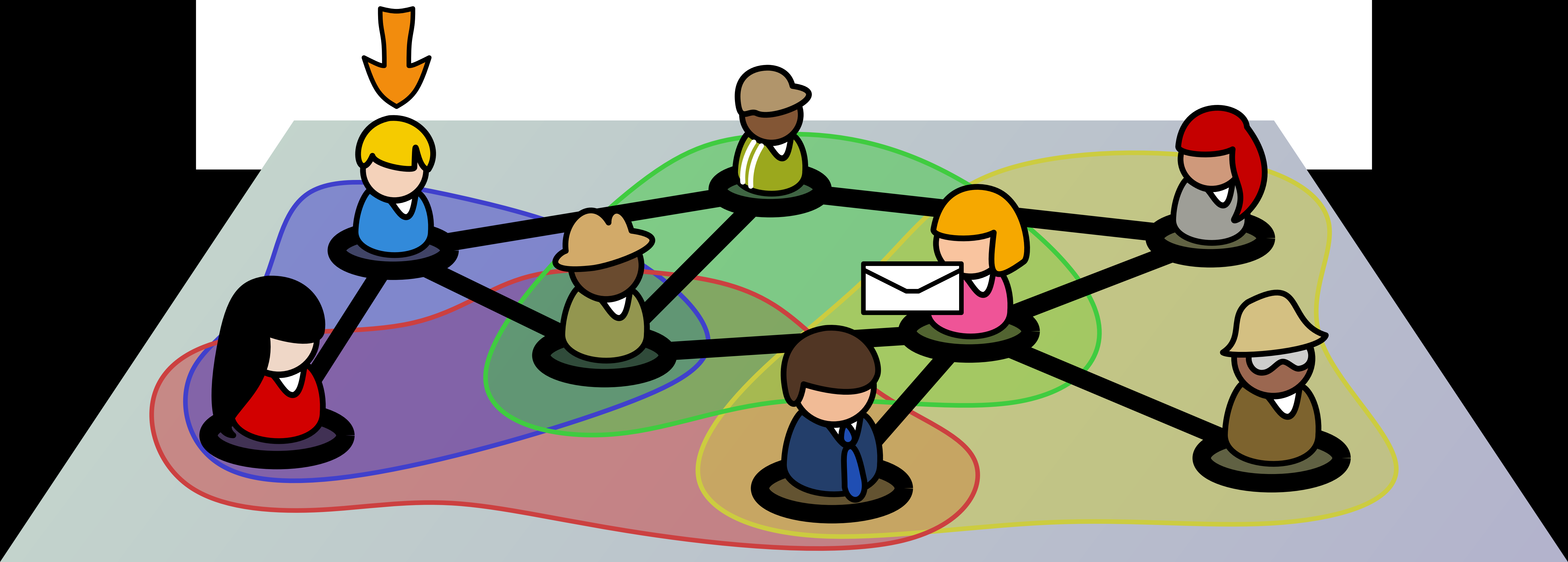

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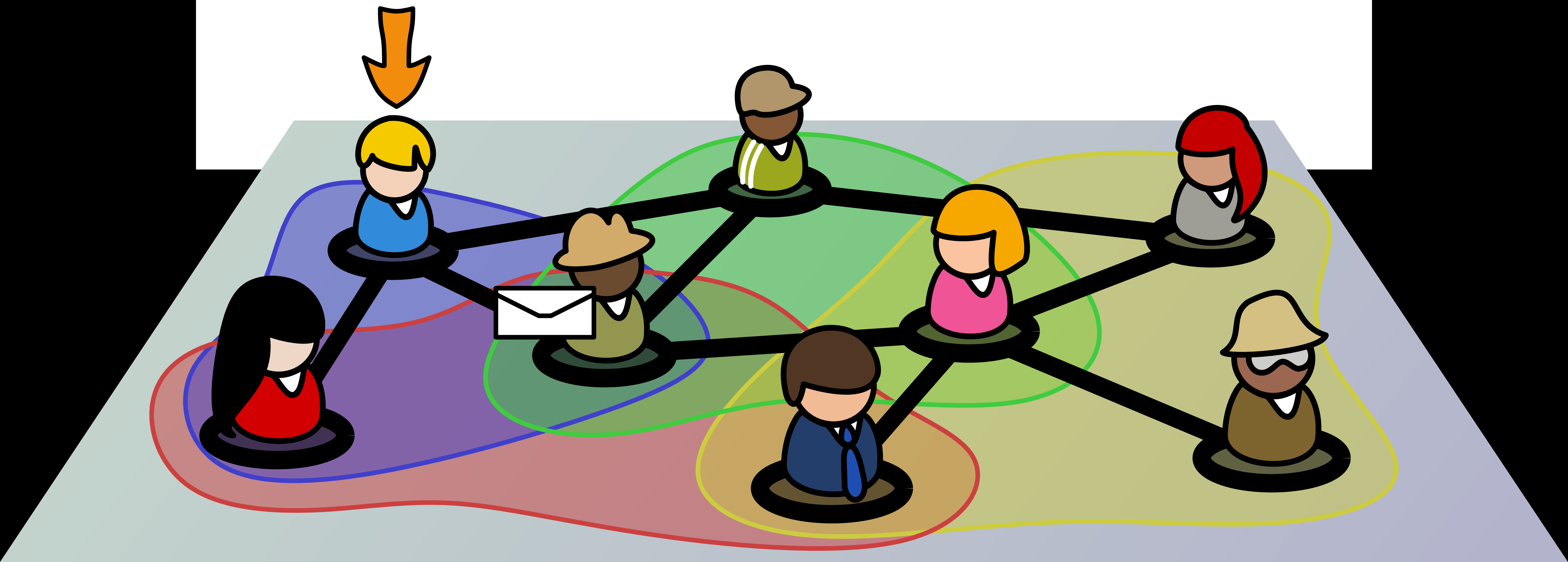

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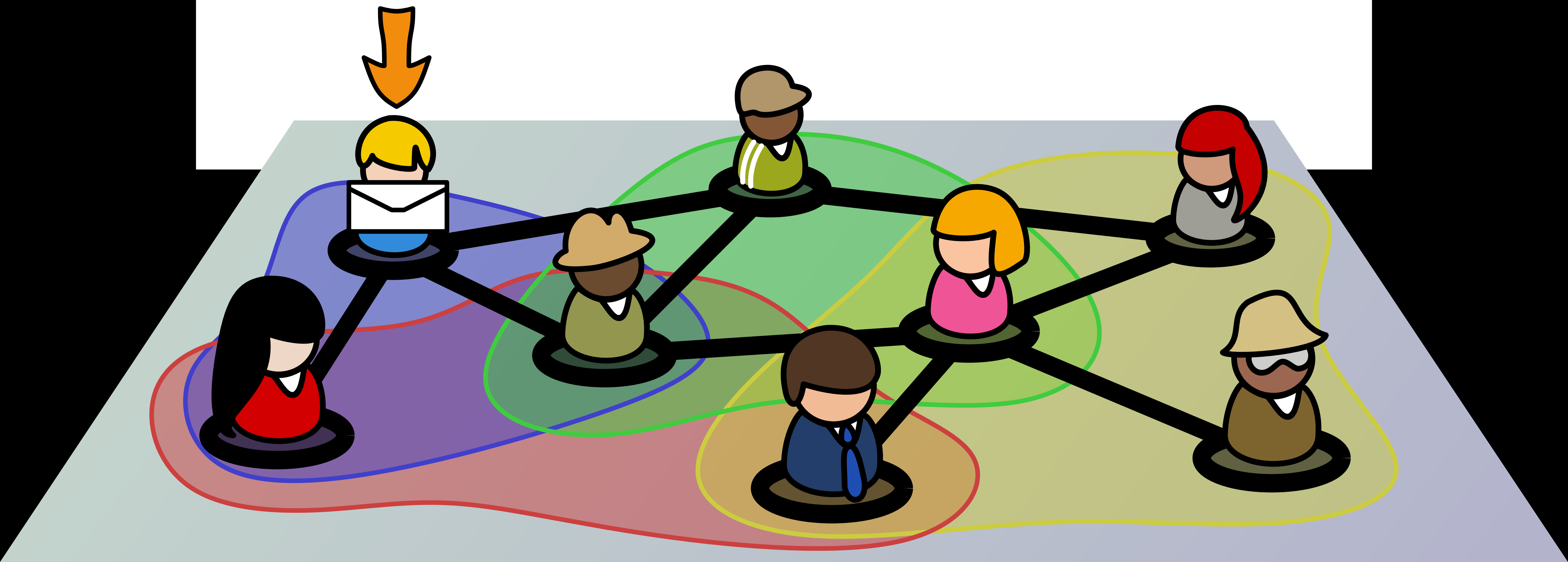

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PART II
DEFINITIONS & RESULTS

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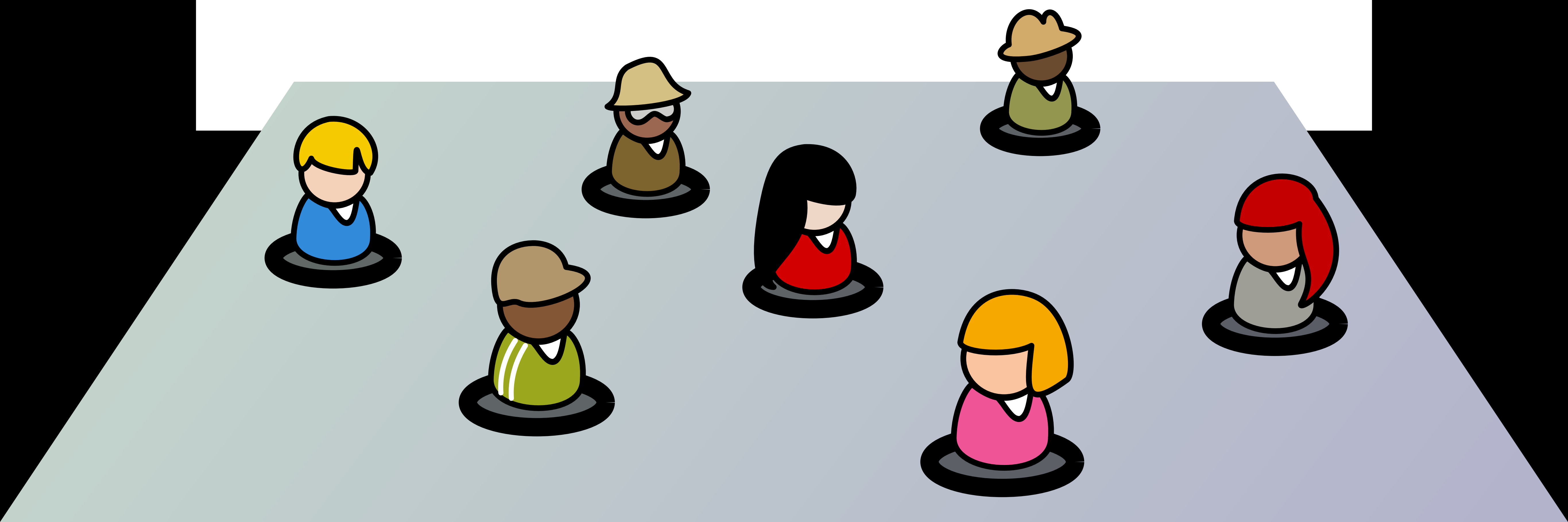

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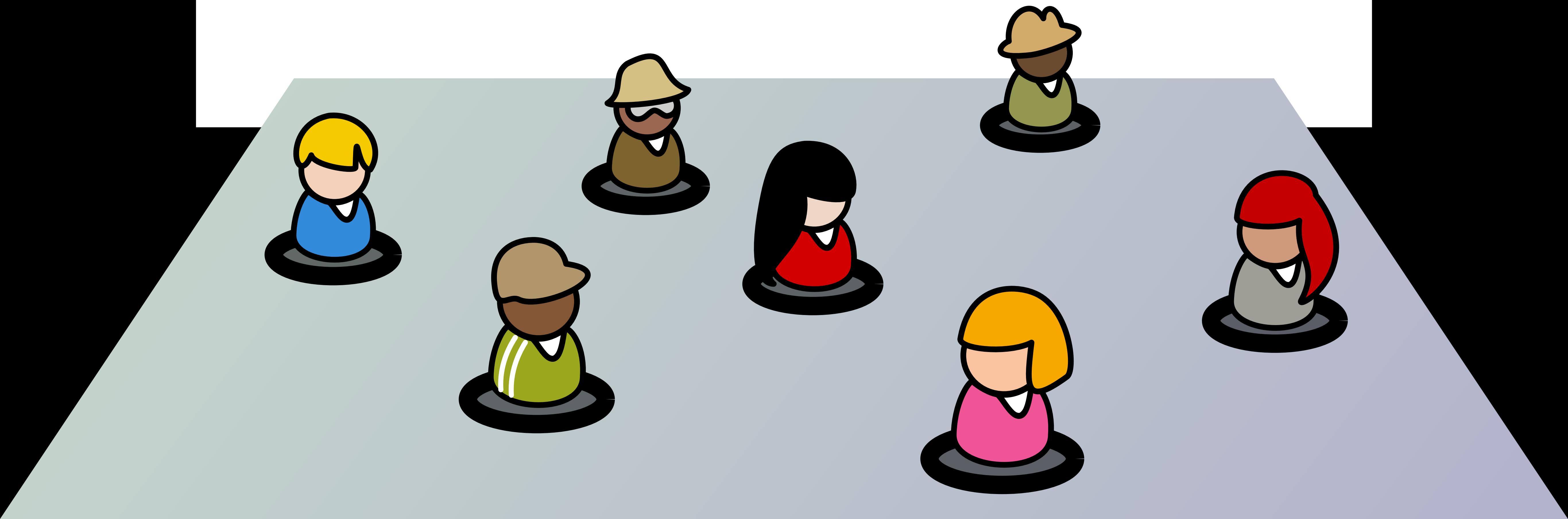

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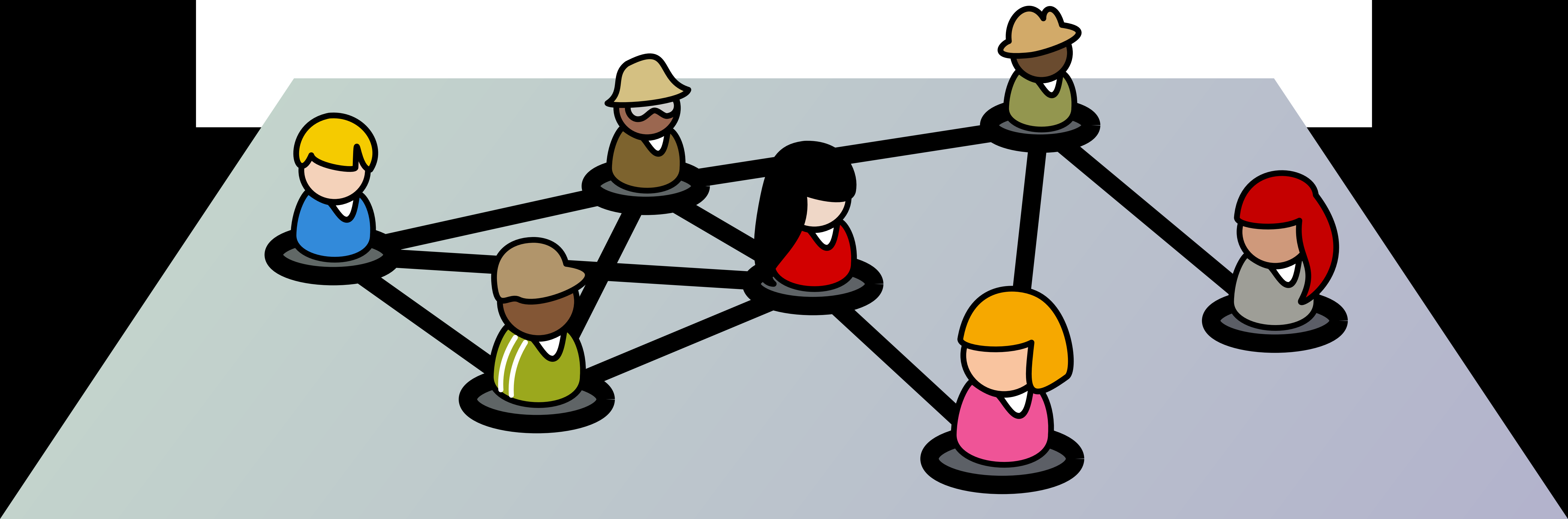

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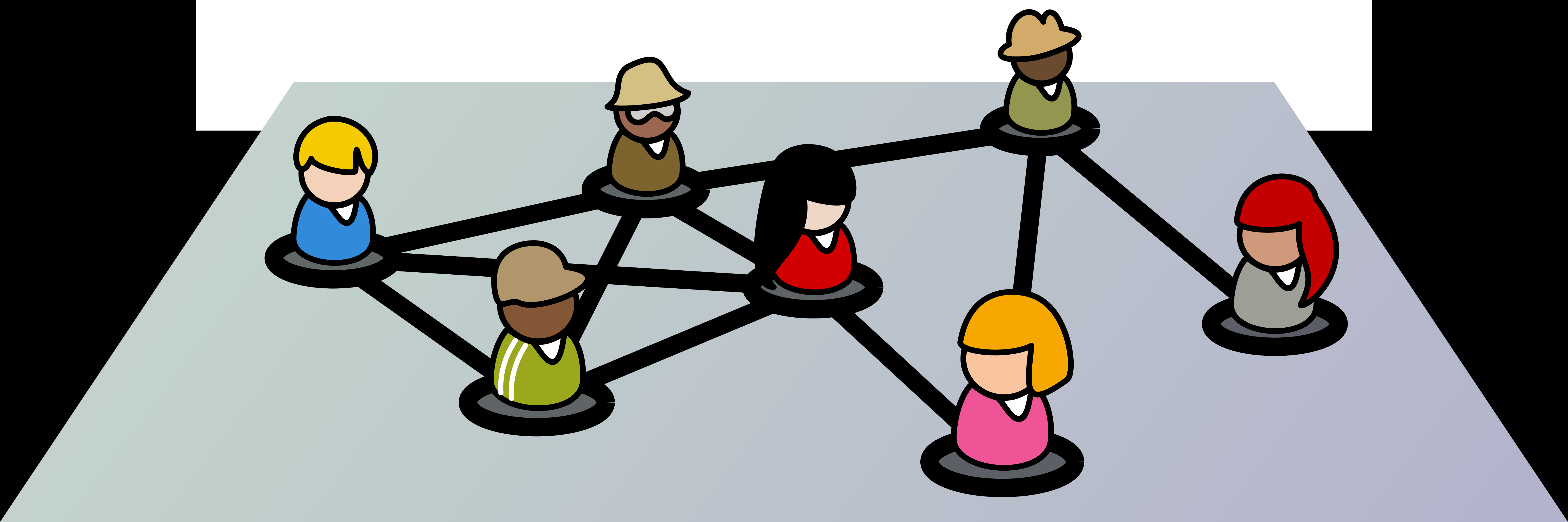

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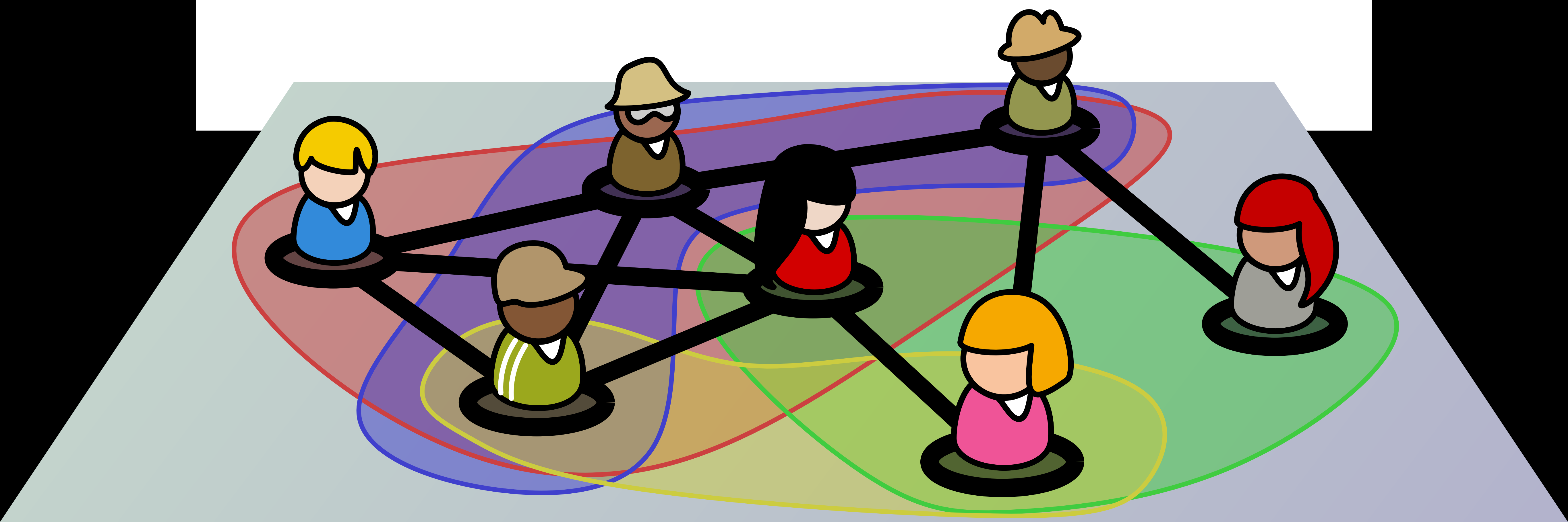

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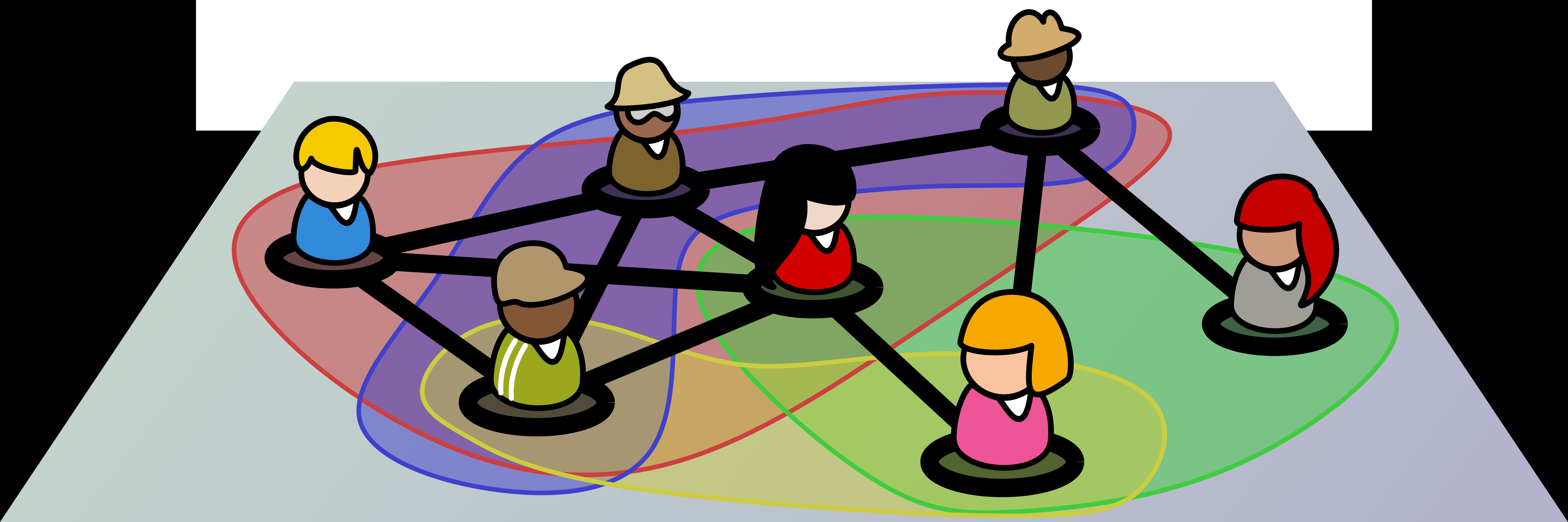

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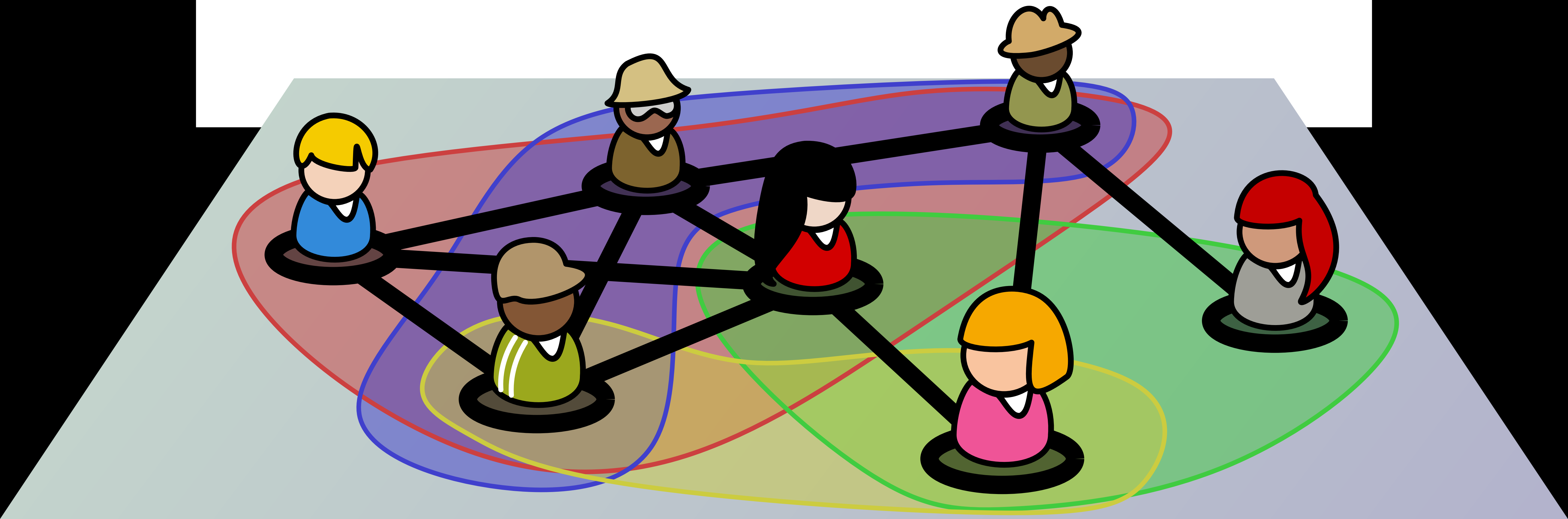

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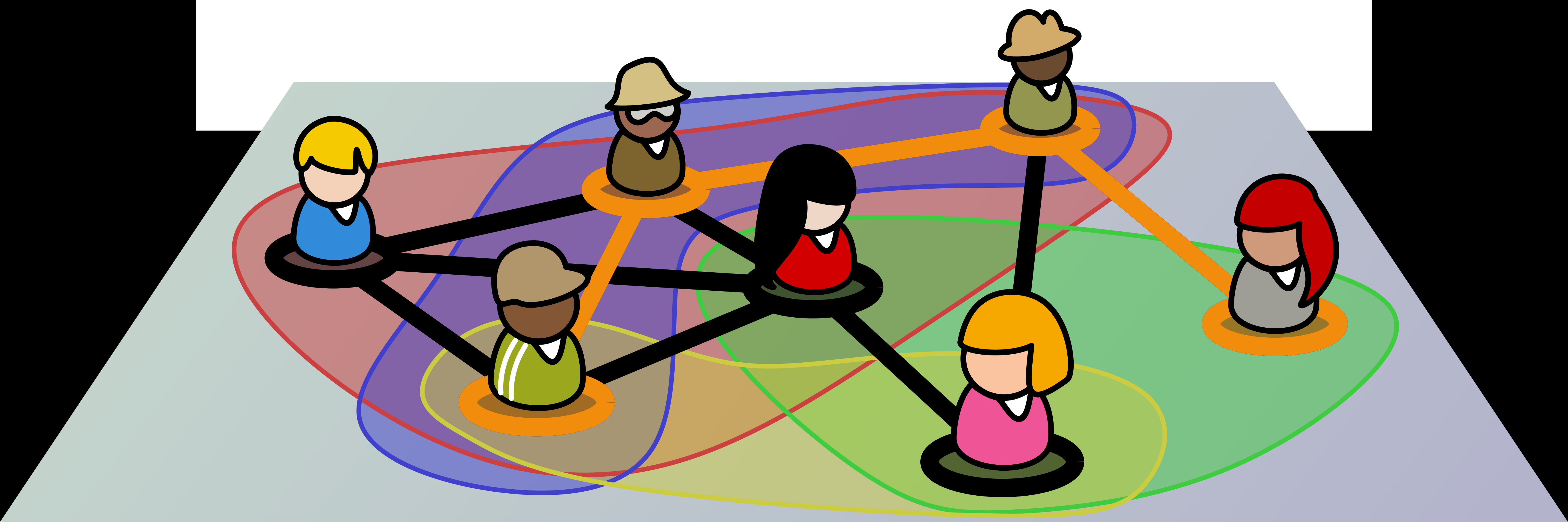

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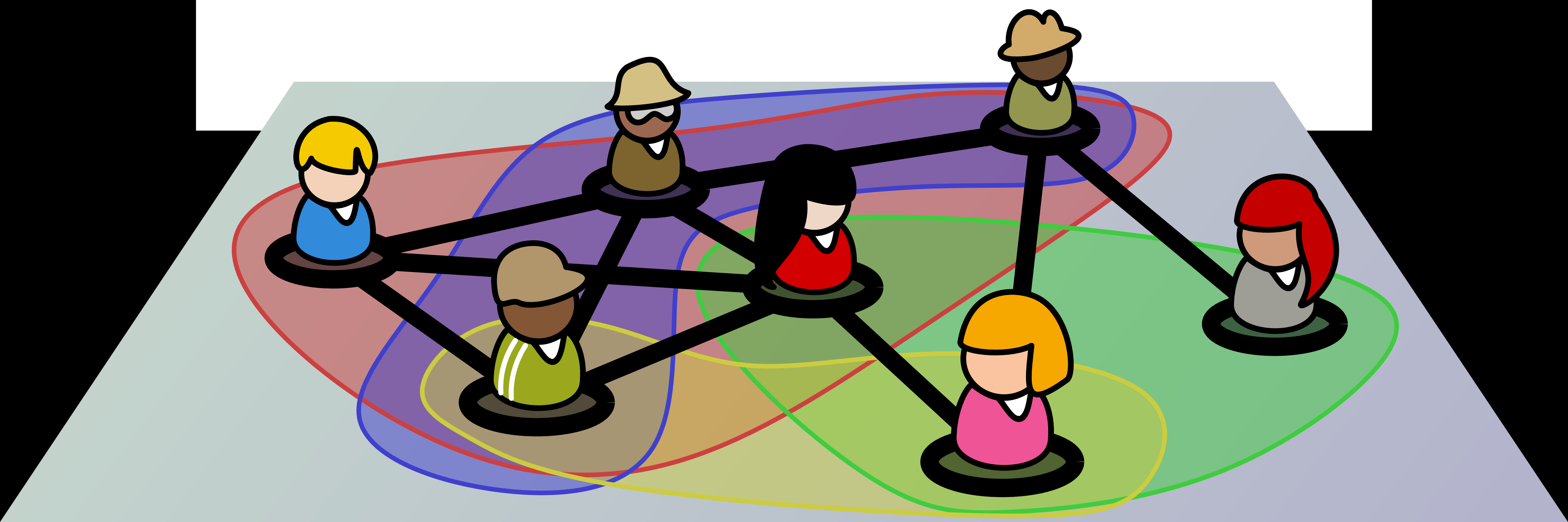

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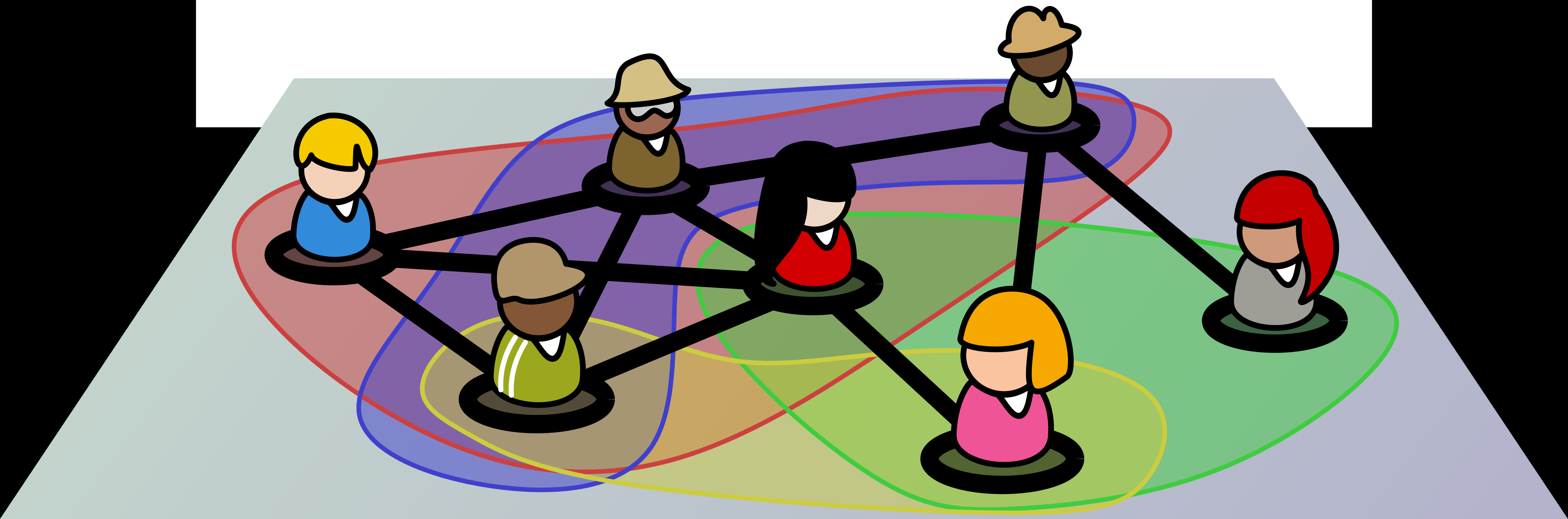

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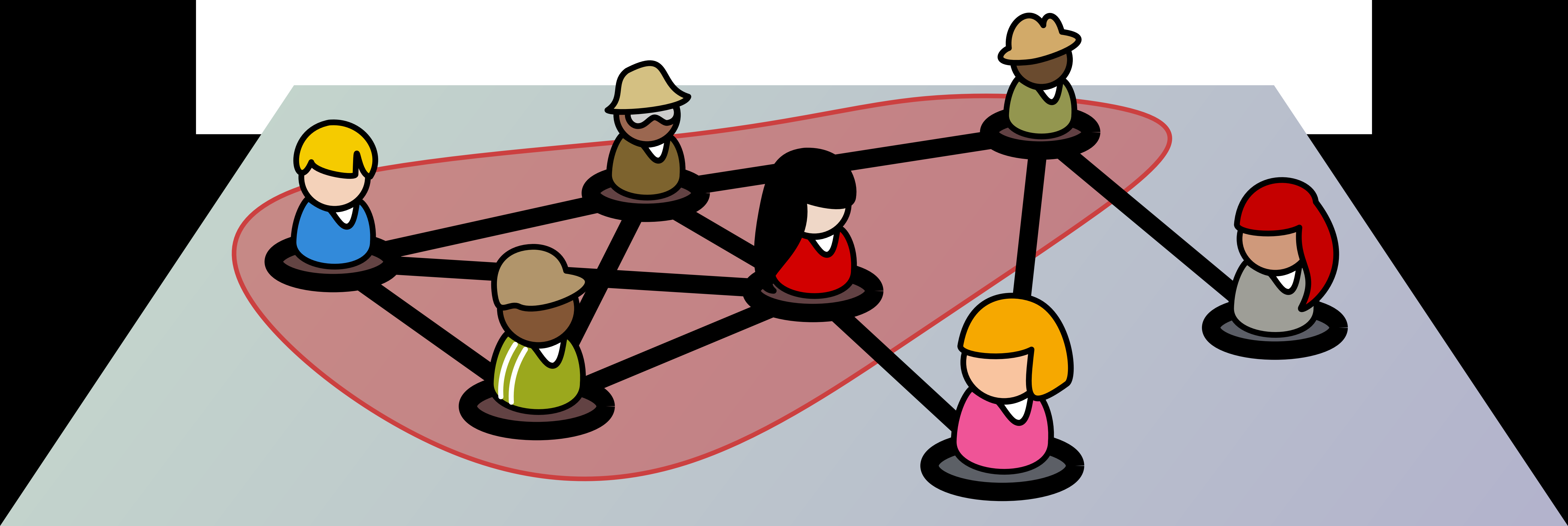

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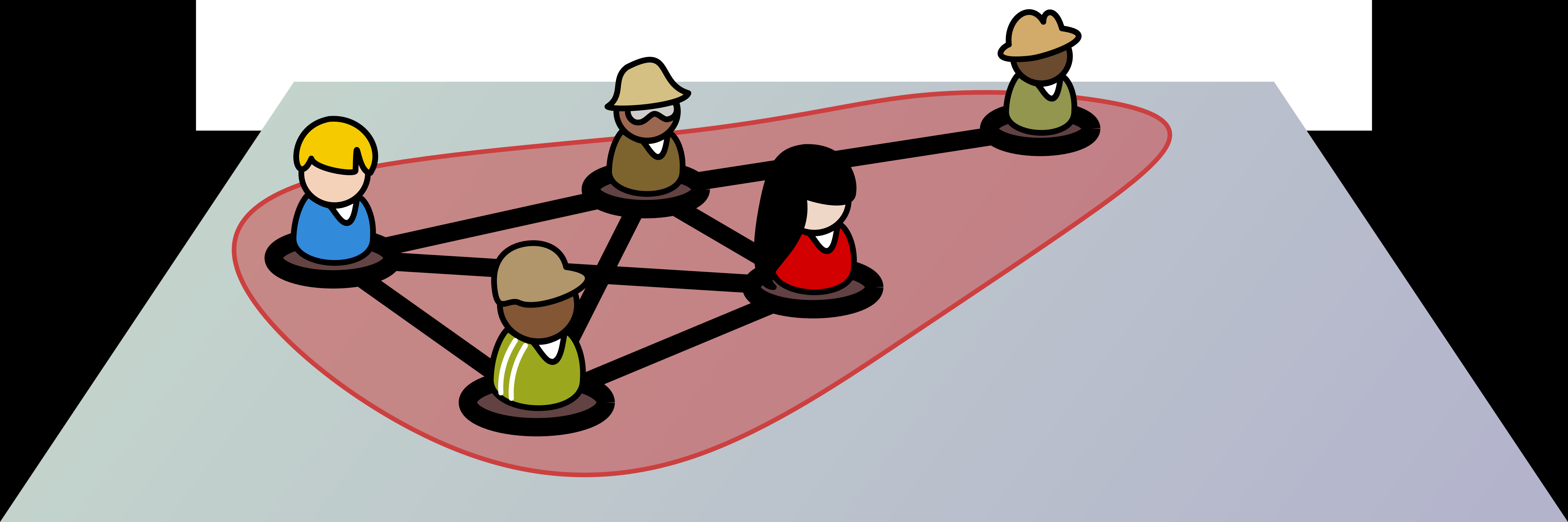

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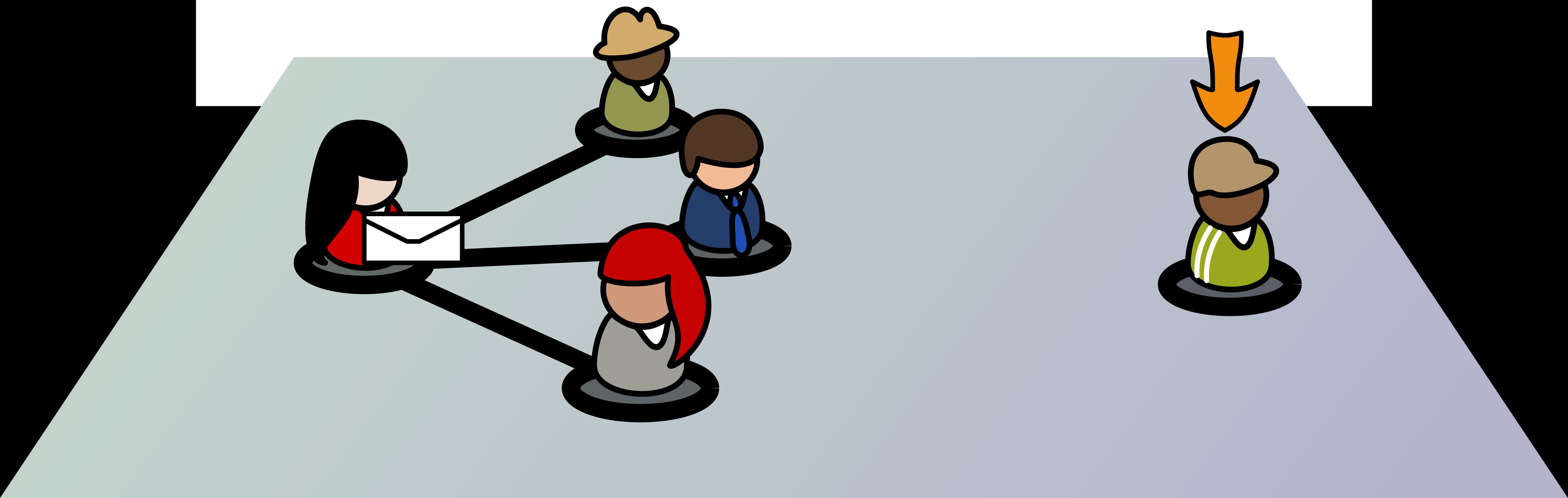

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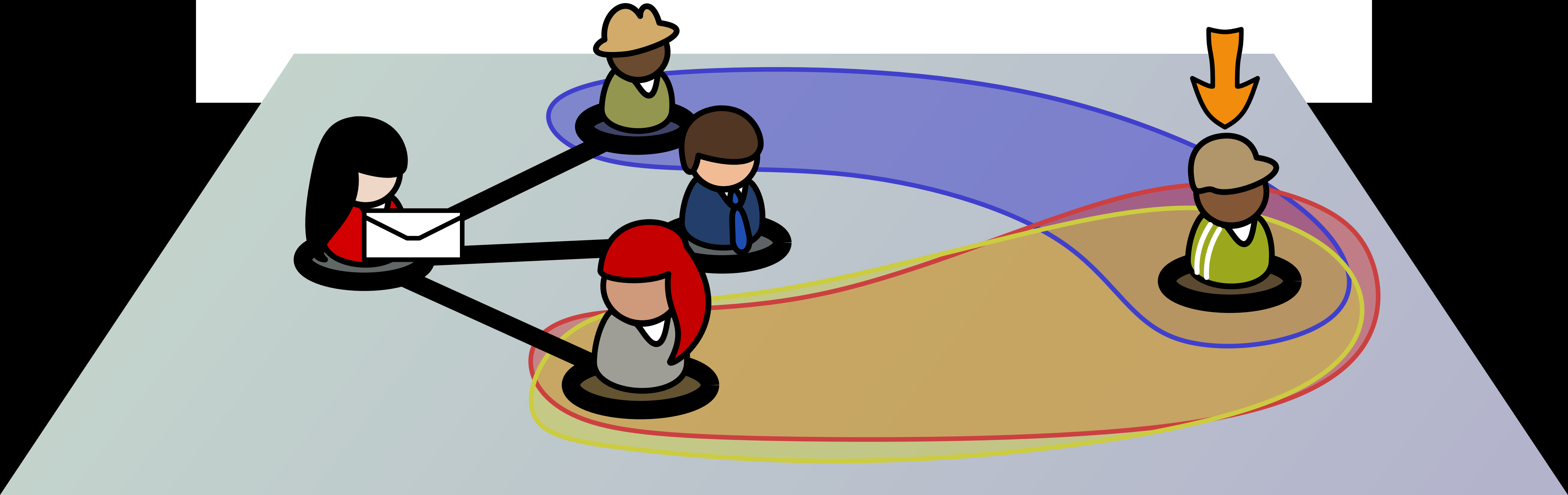

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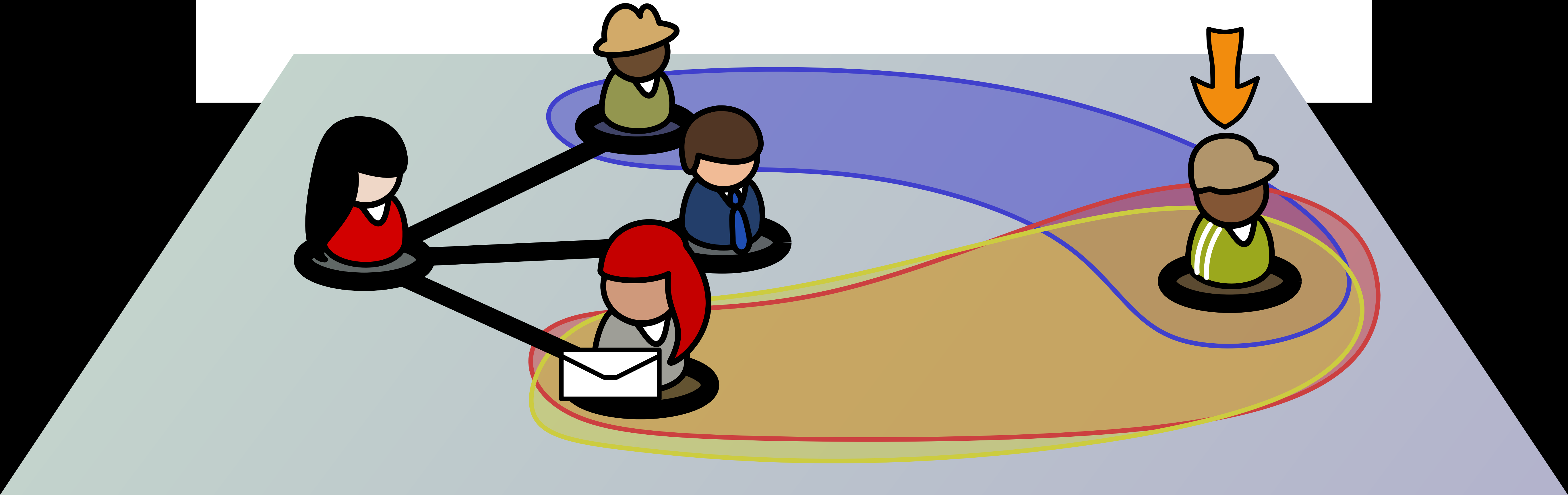

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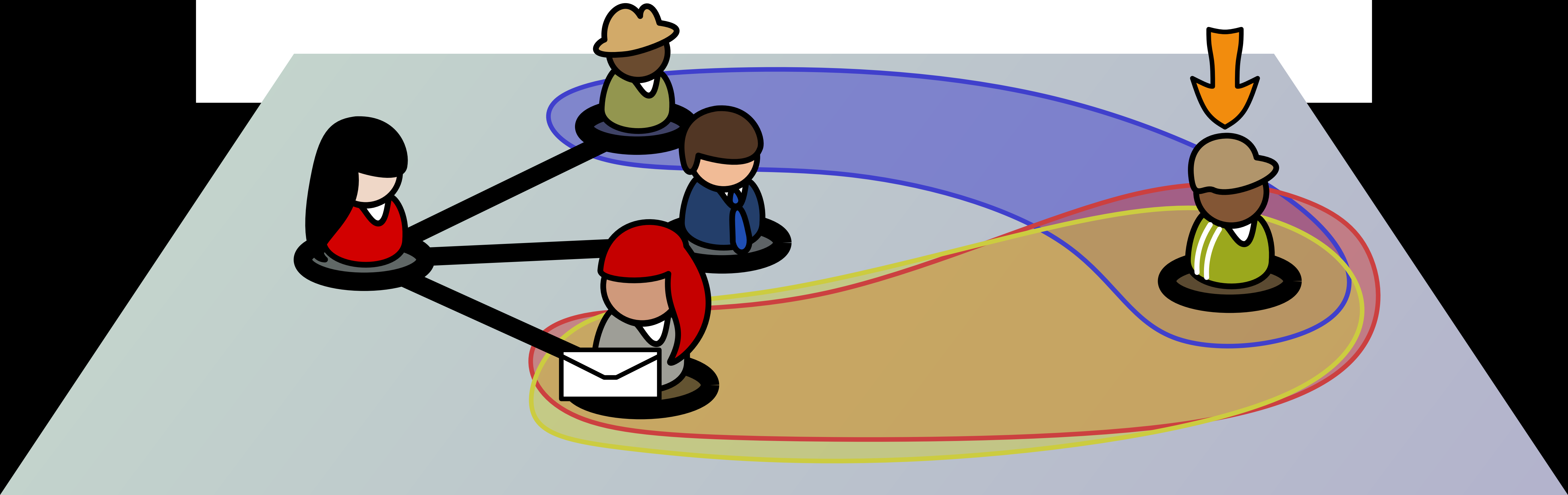

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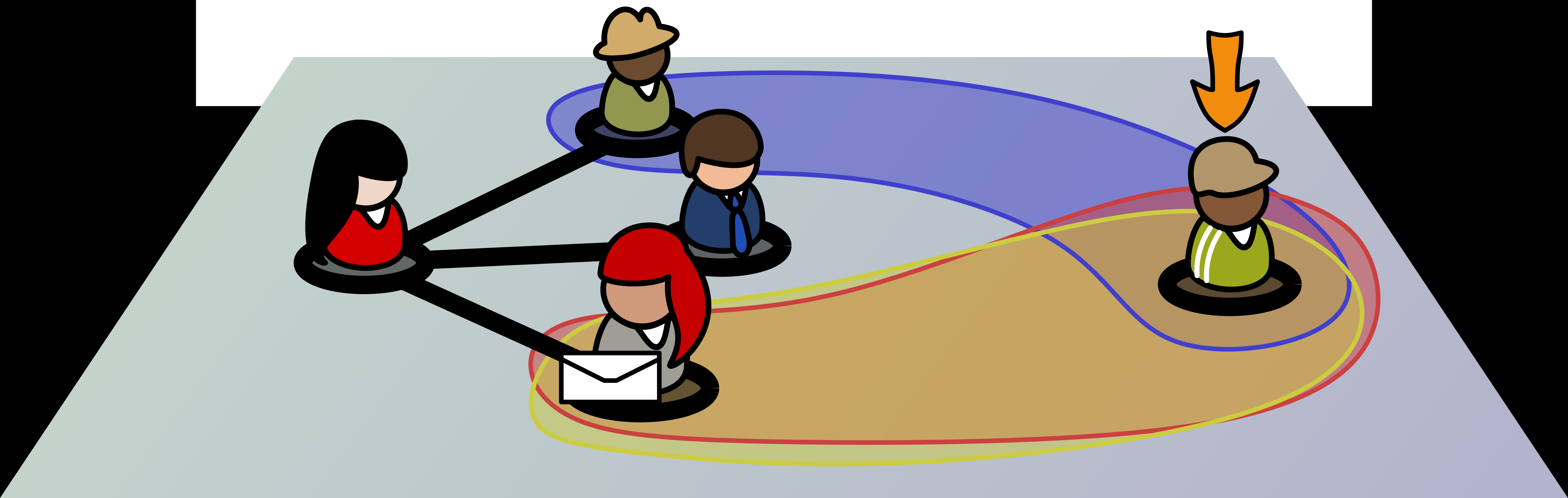

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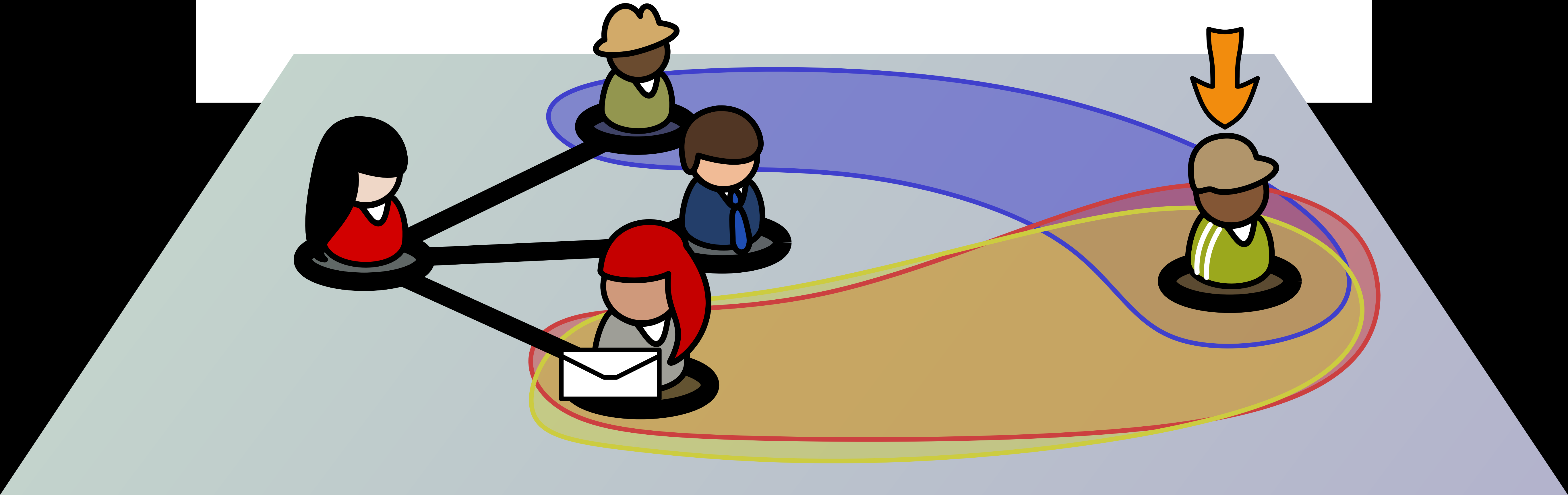
  - Requires only local knowledge about neighbours and target



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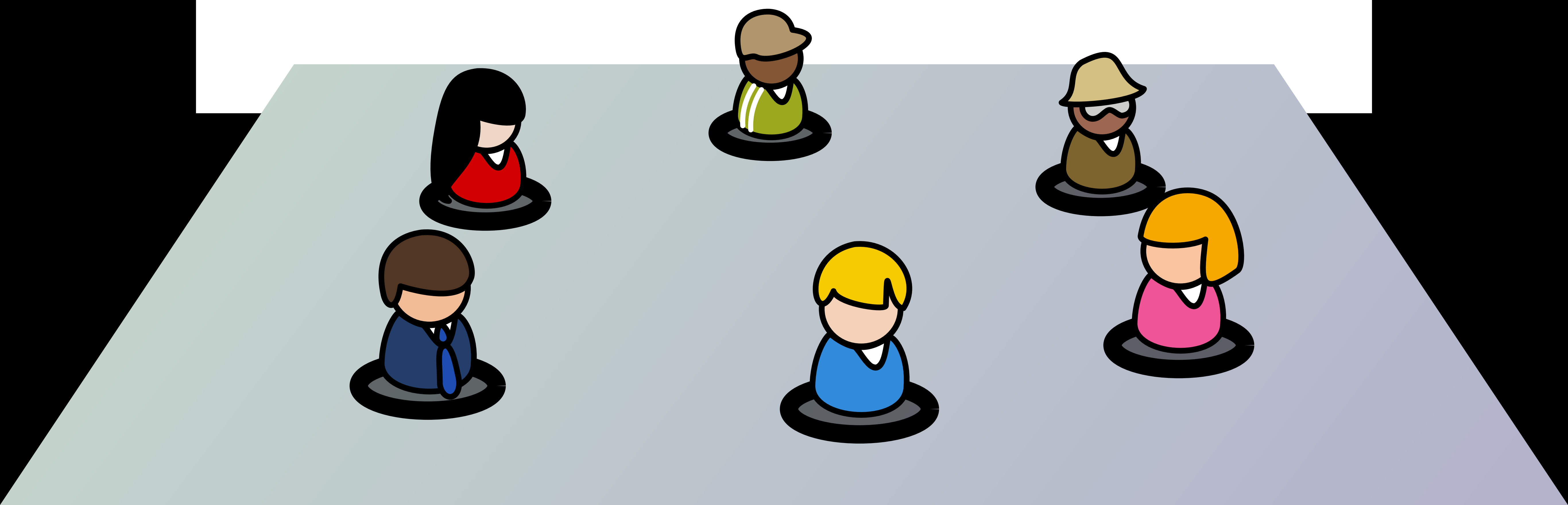

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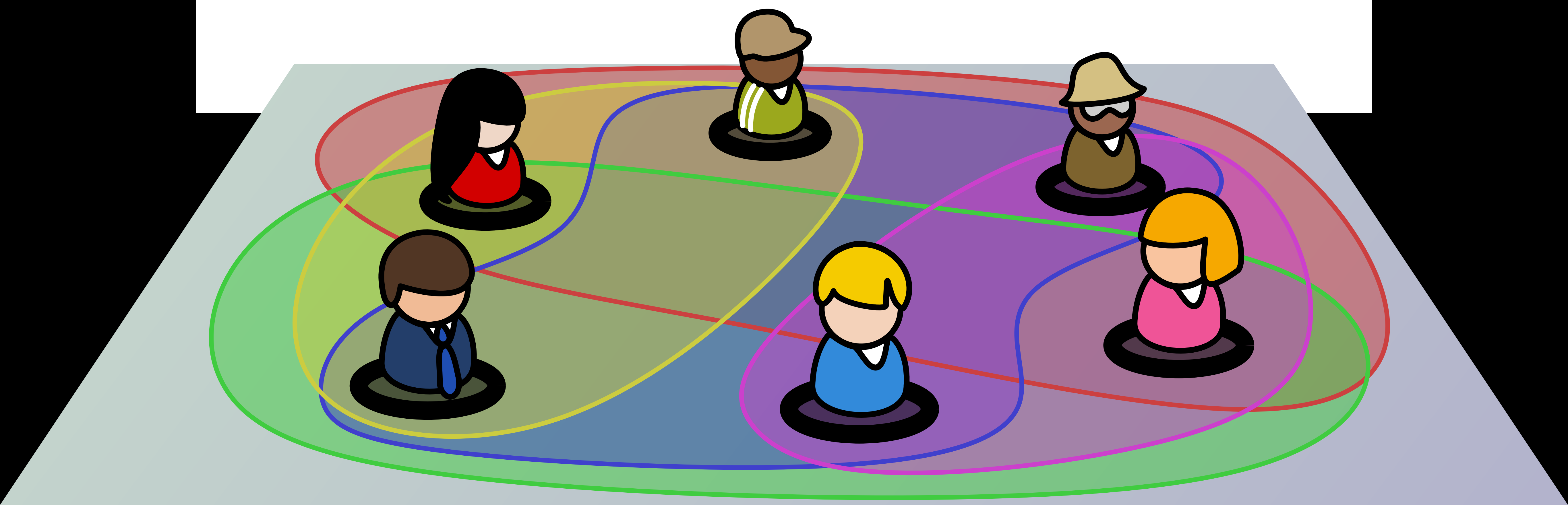

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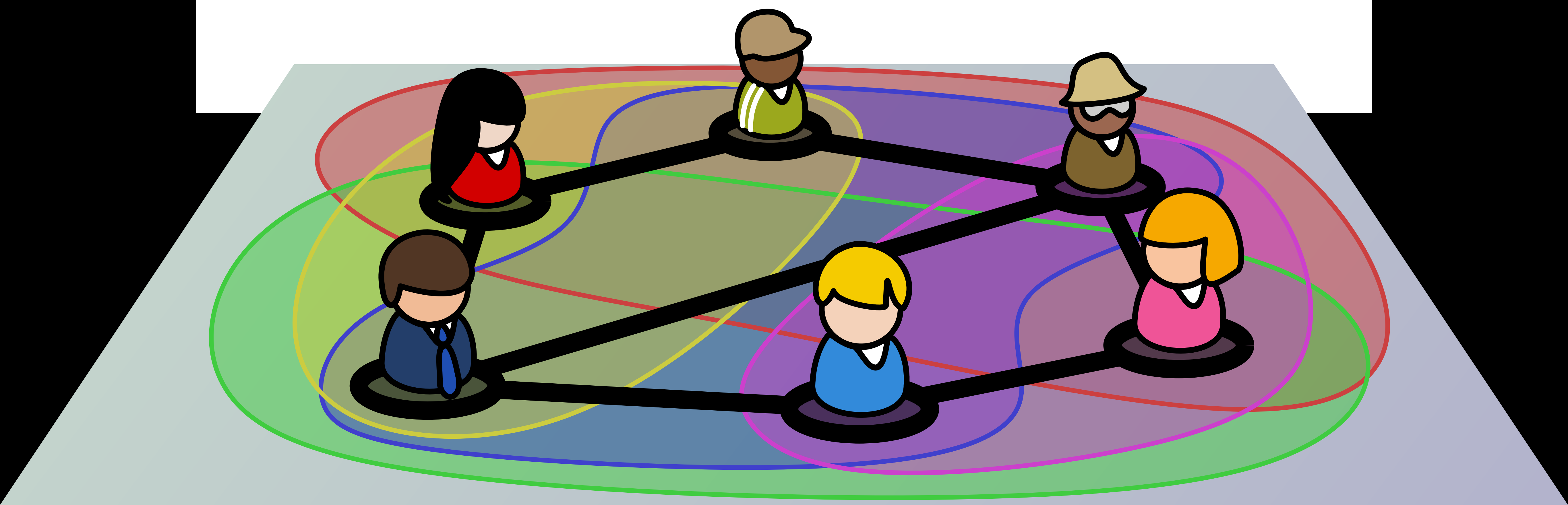

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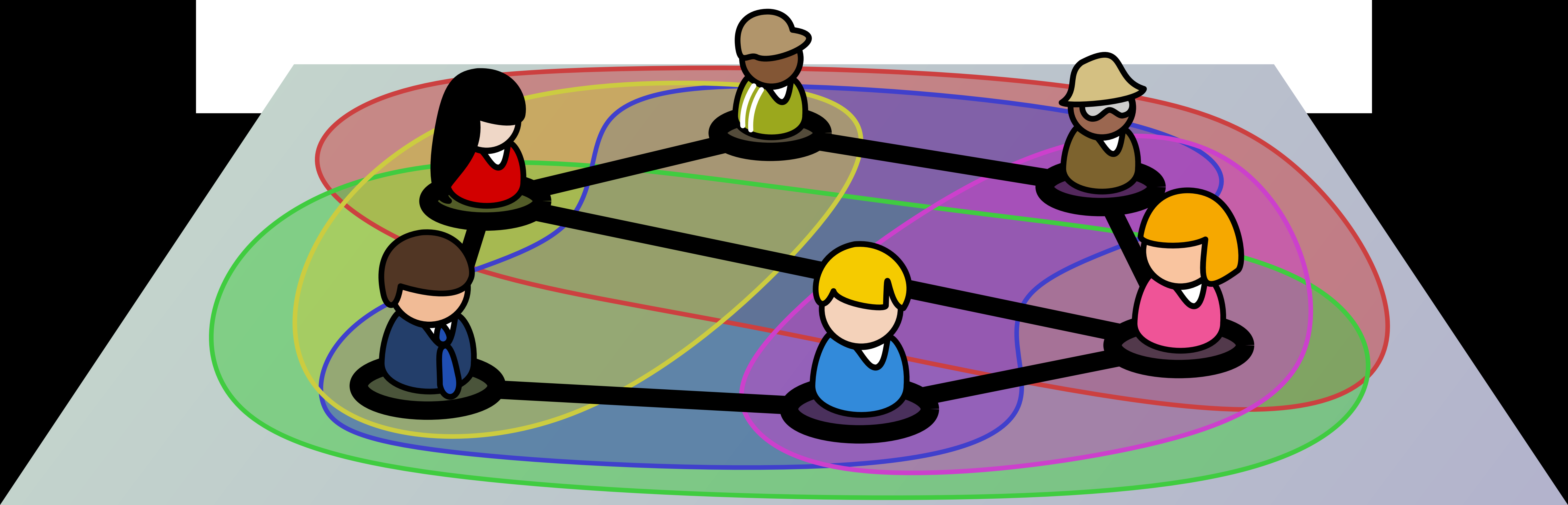

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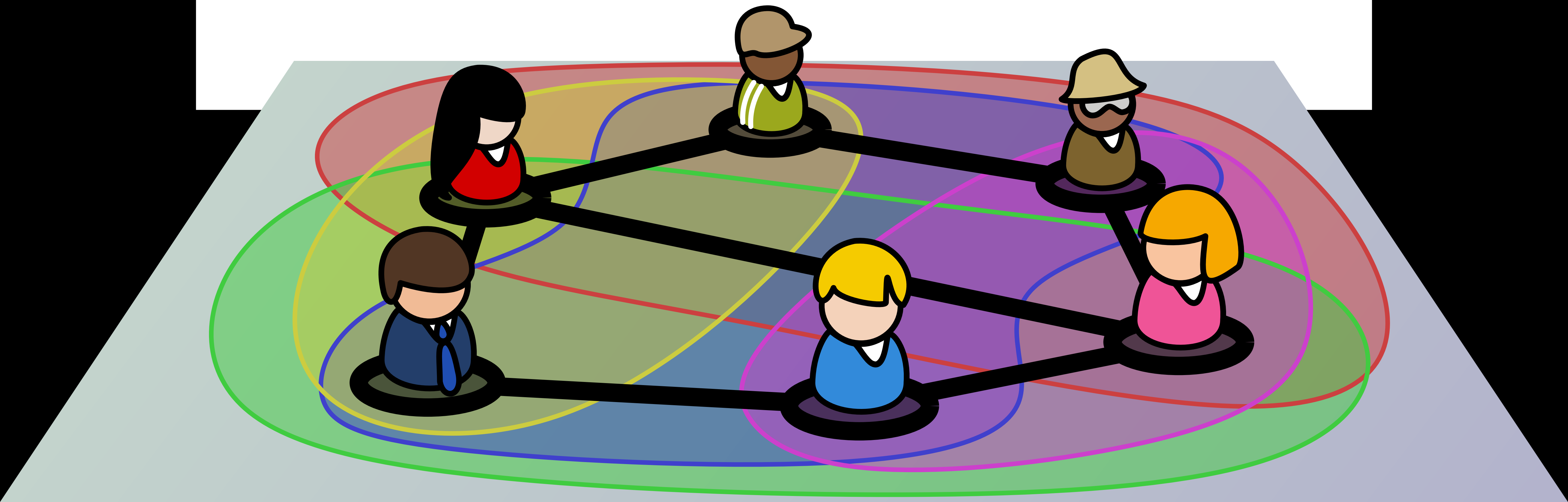

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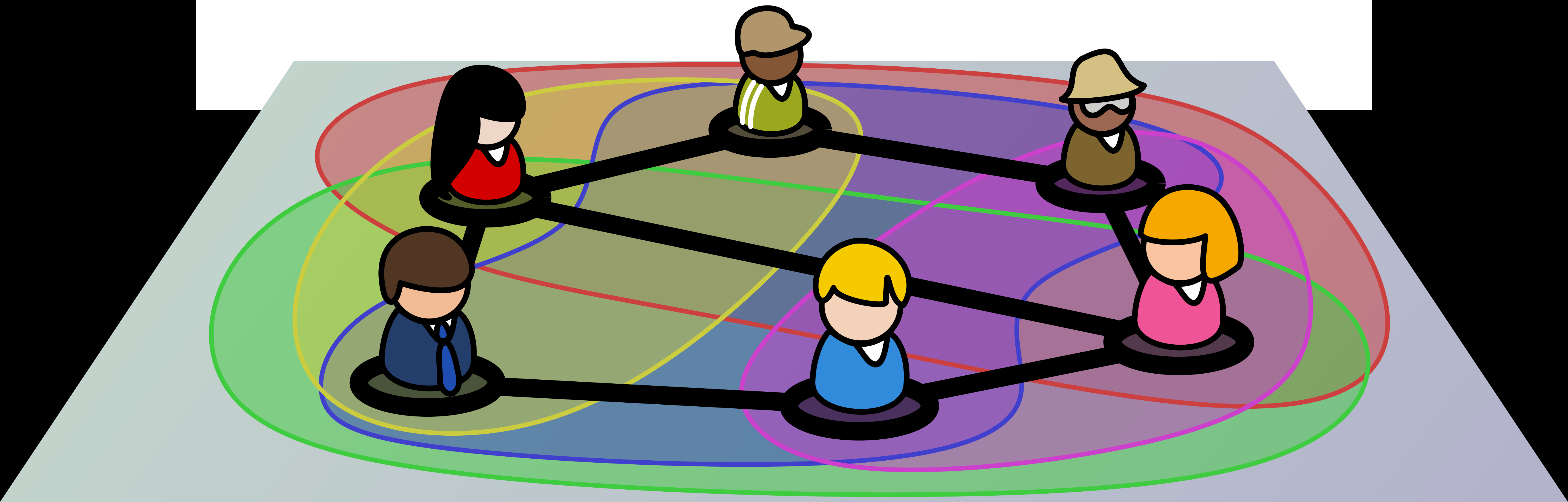

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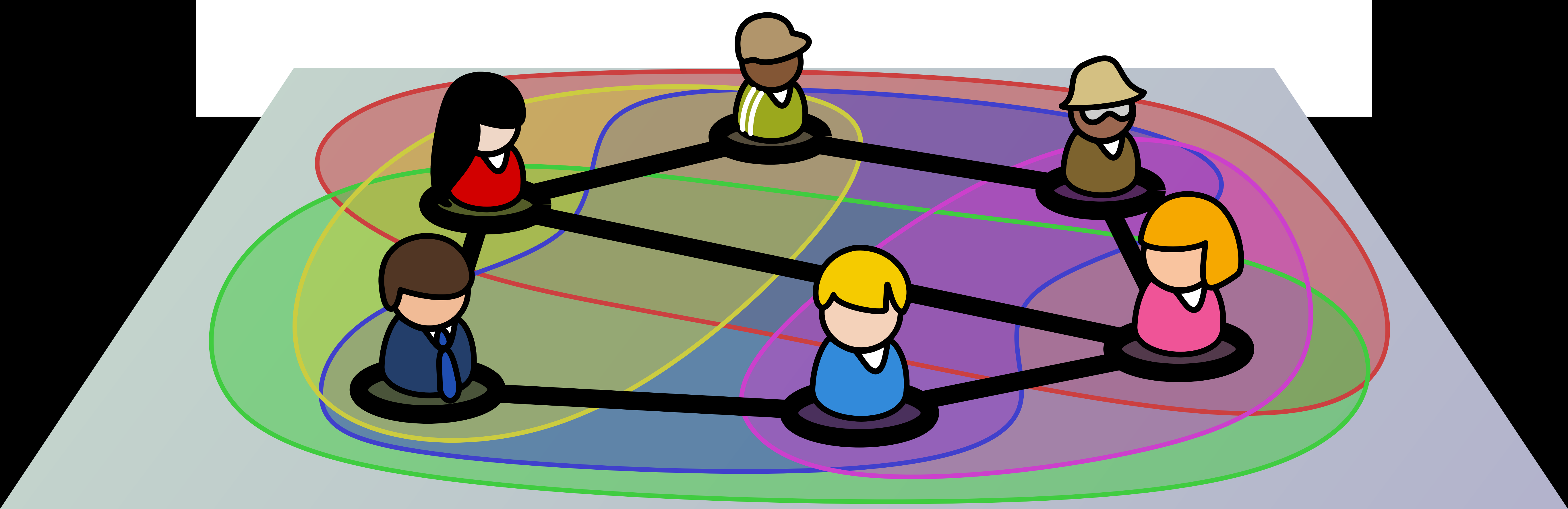
  - Seems like a natural assumption
  - Makes it a lot easier to reason about simple routing



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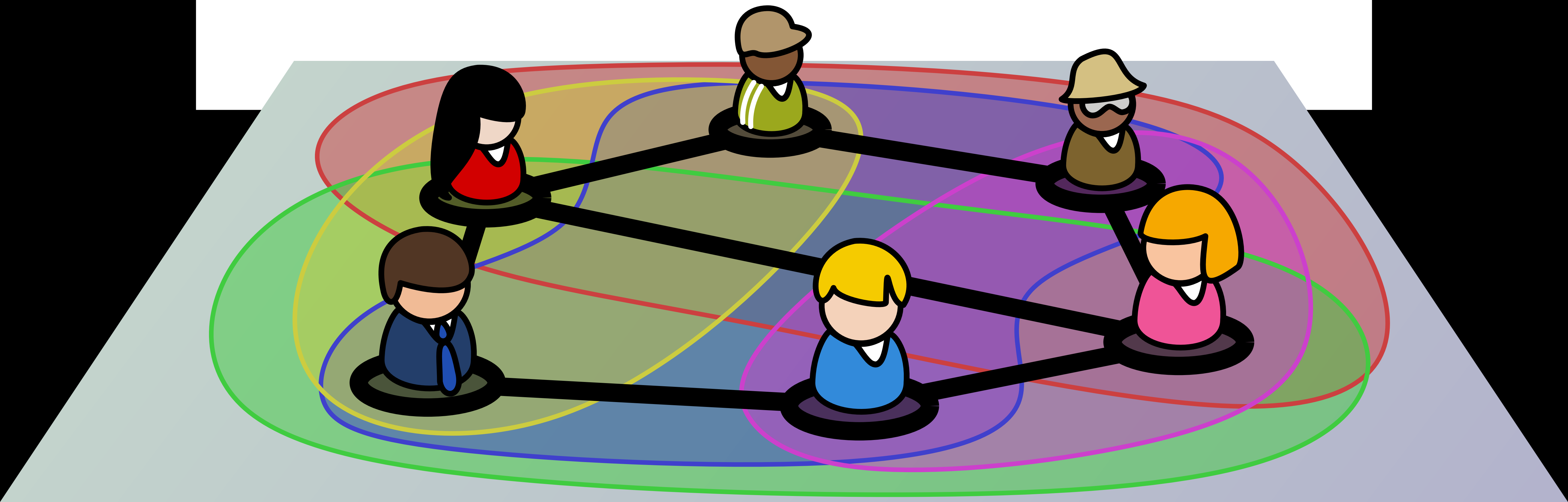

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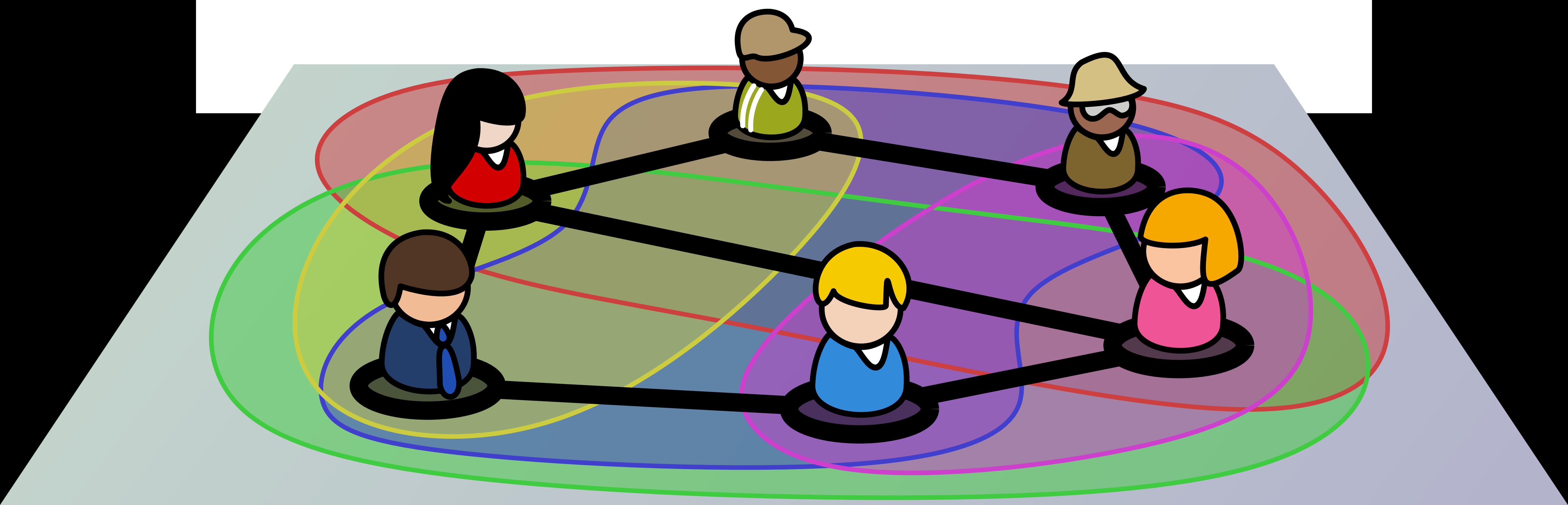

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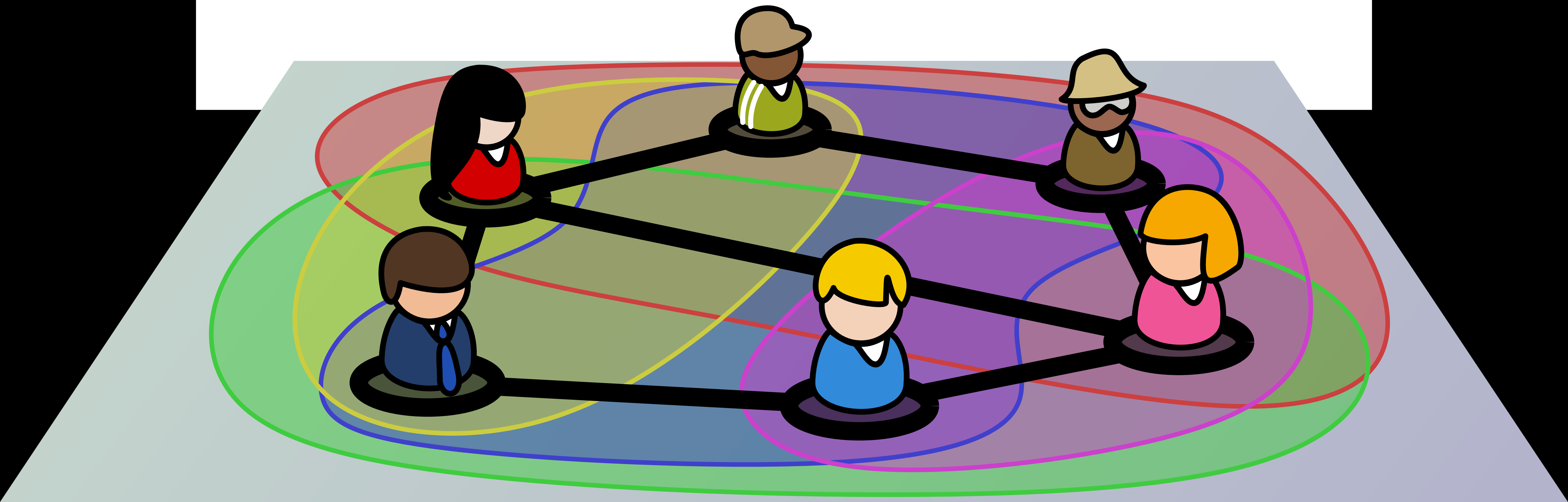

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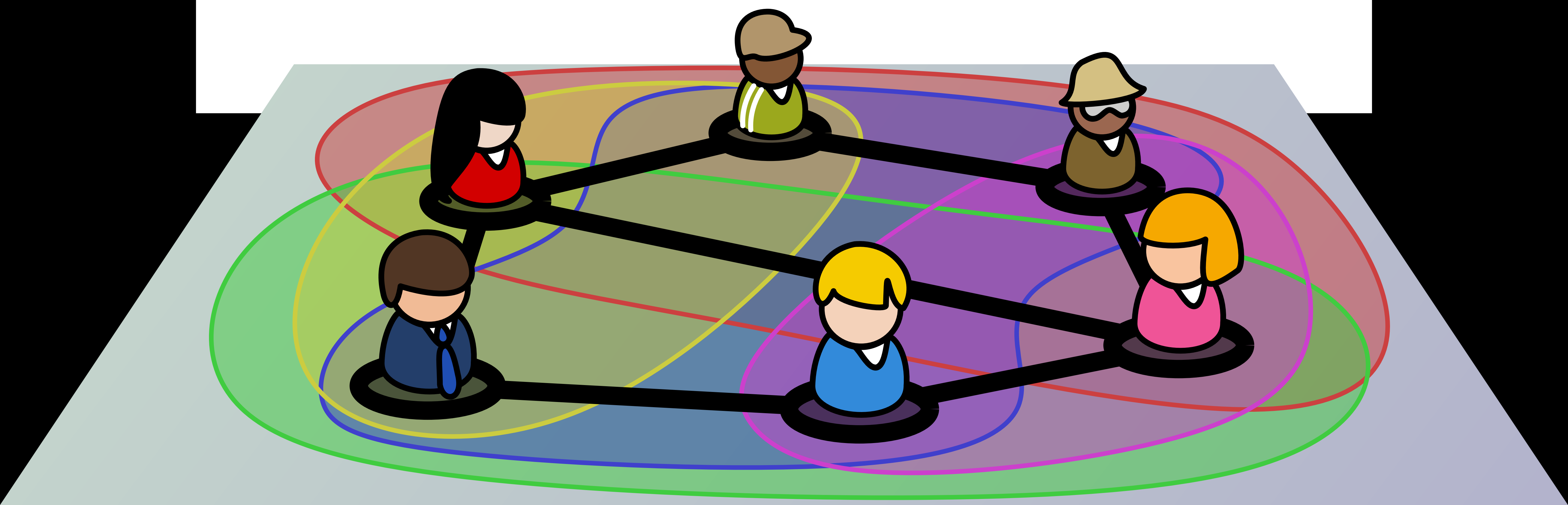

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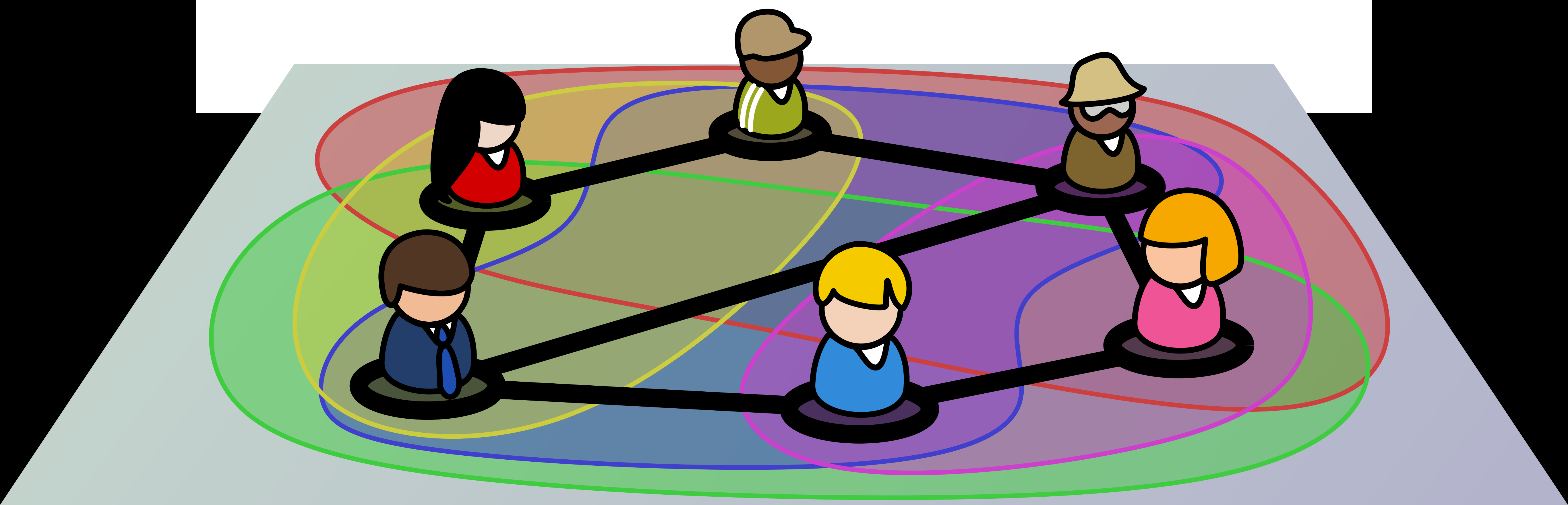

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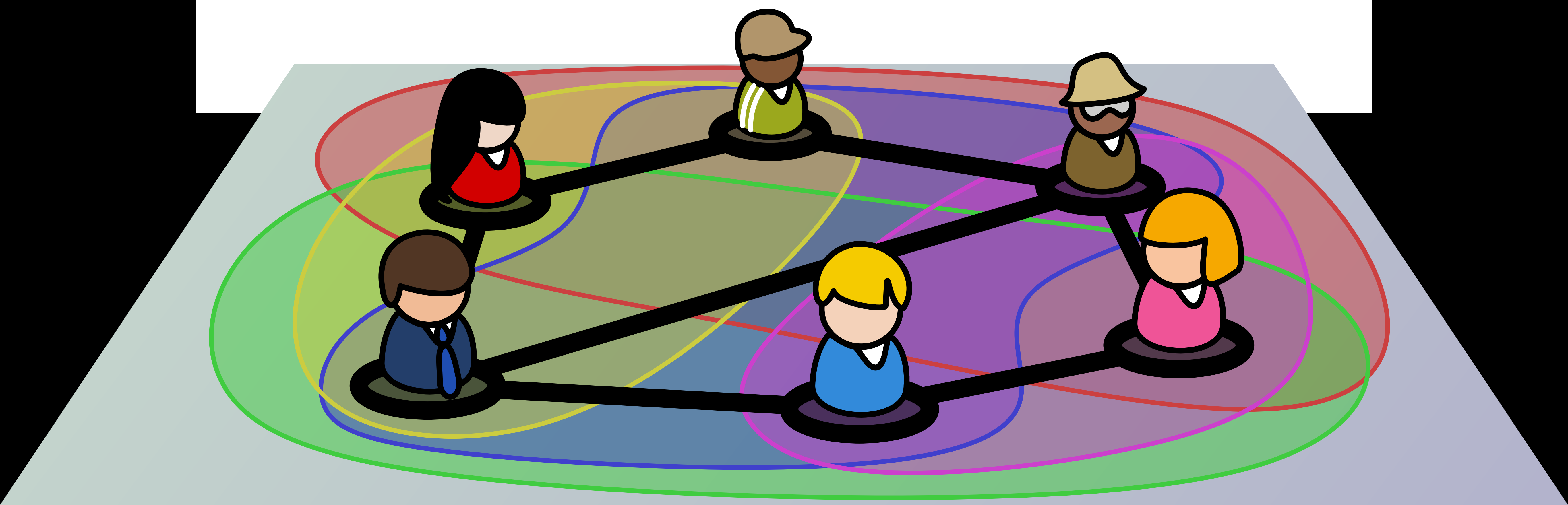

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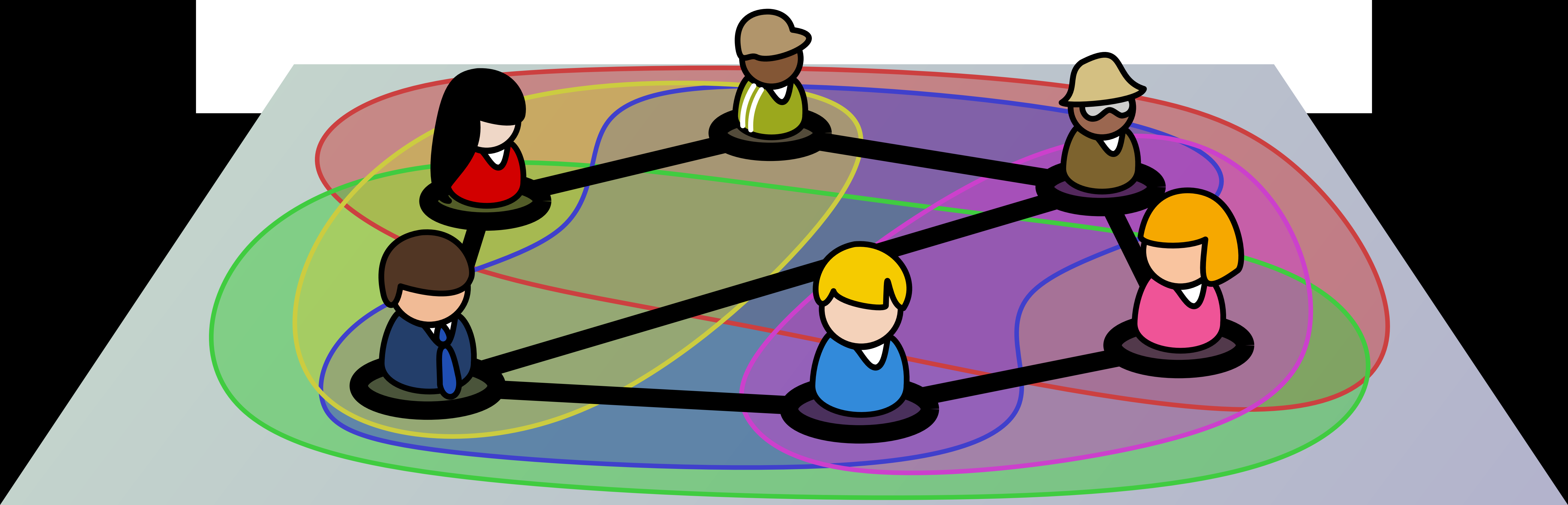

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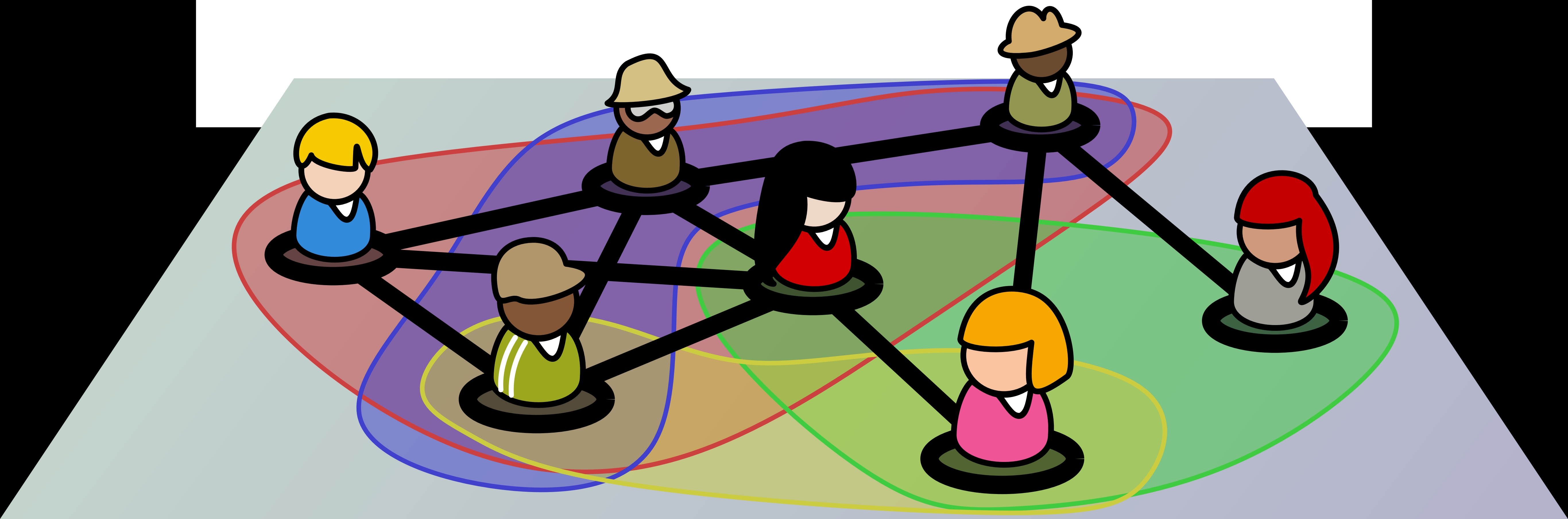

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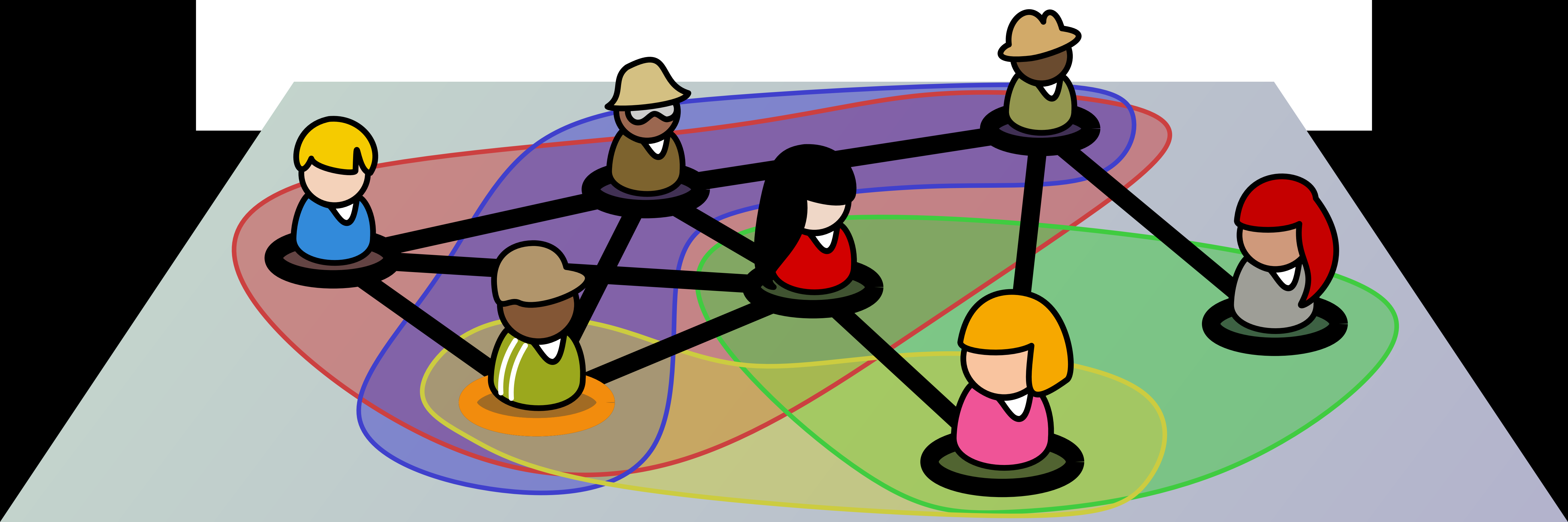

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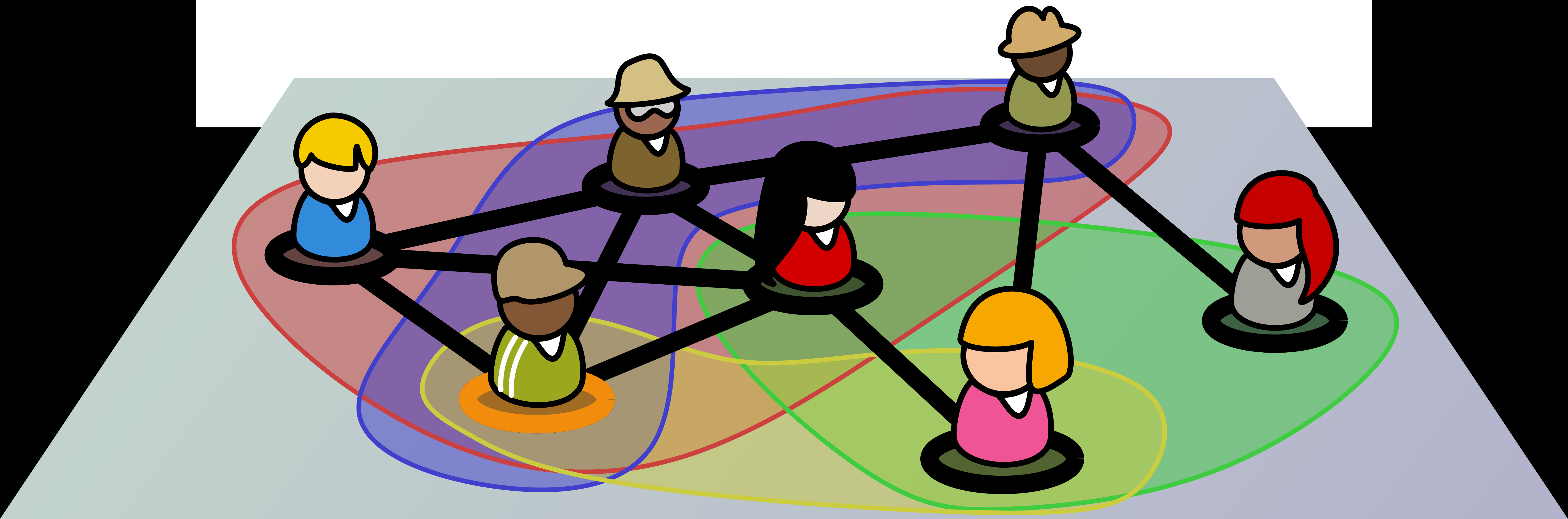

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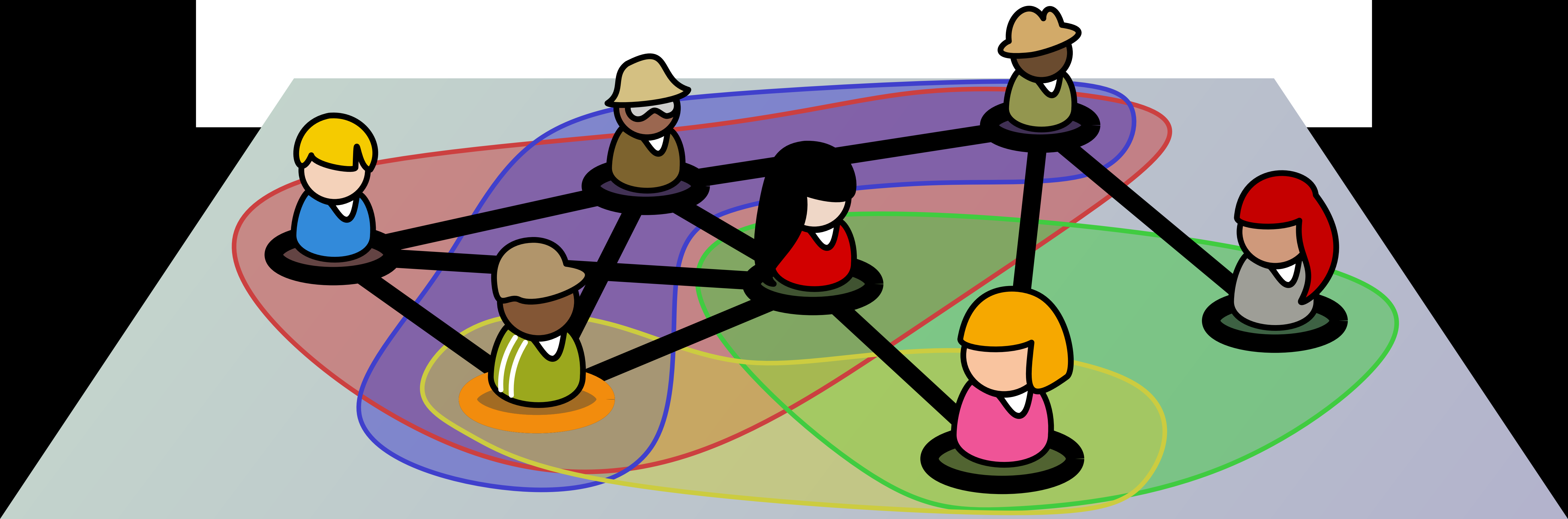

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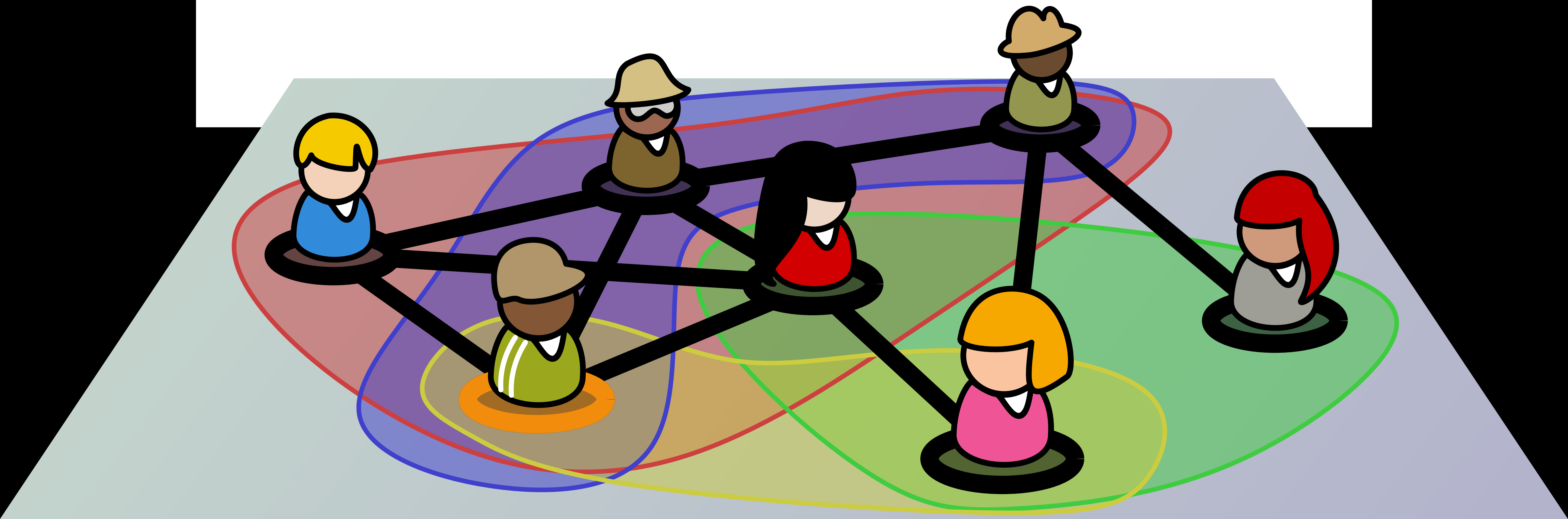
- Rationale
  - Captures the "cognitive load" of people
  - We expect the membership dimension to be *small*



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  - $\forall \mathcal{S} \exists G$: no

PART III
TECHNICAL DETAILS

CATEGORY CONSTRUCTION

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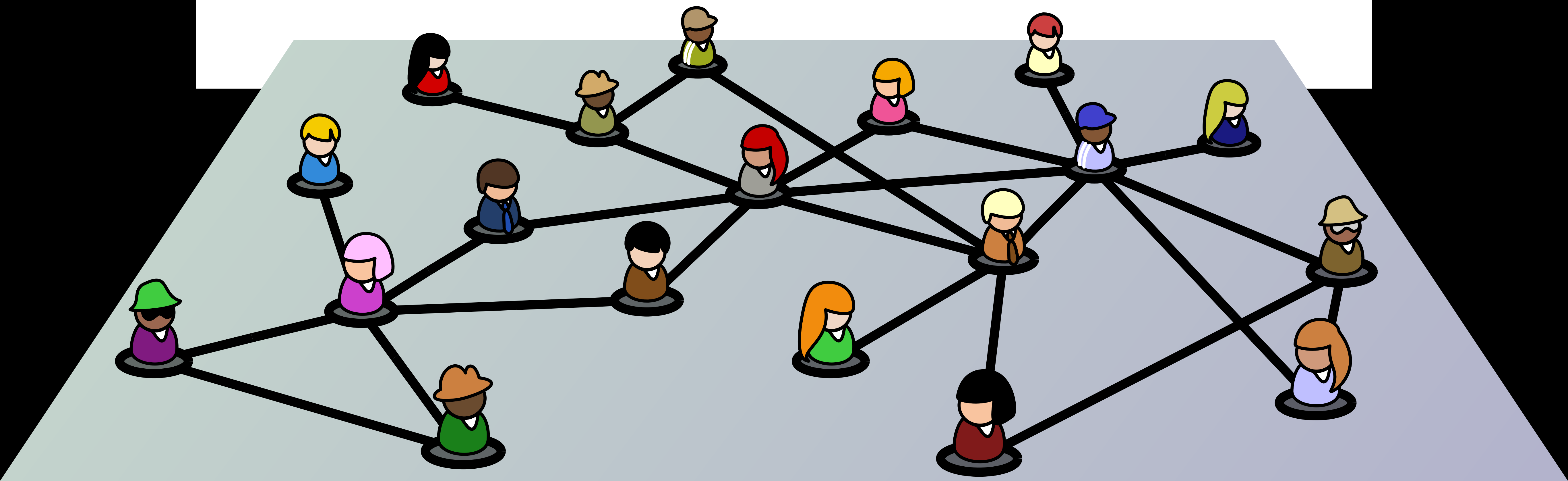

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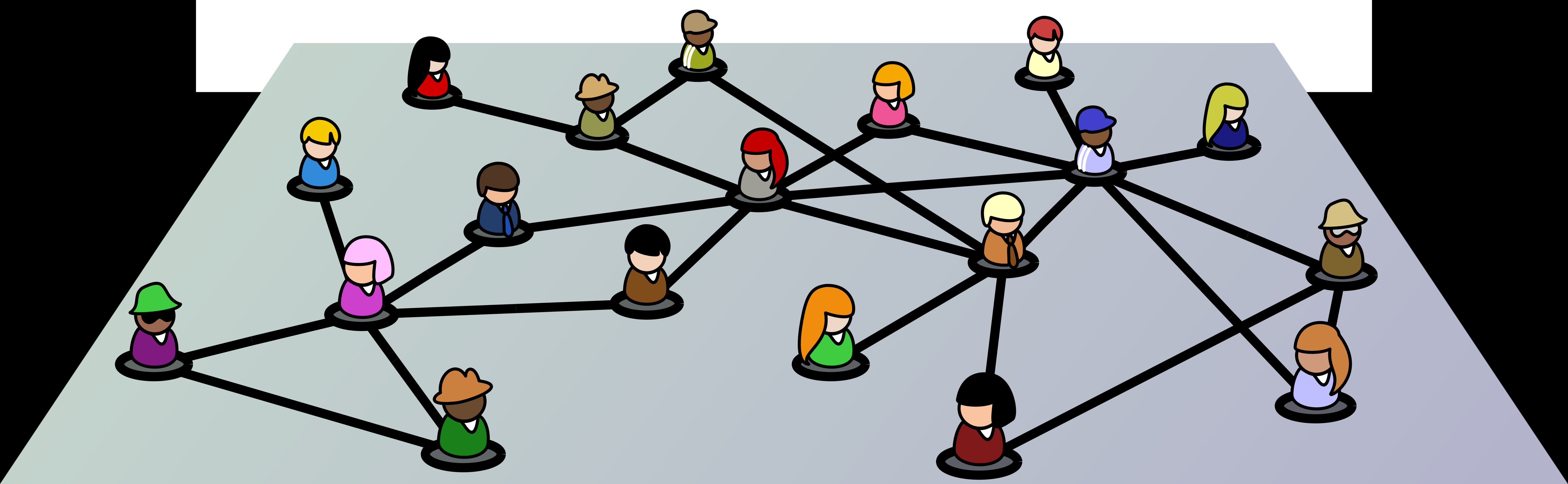

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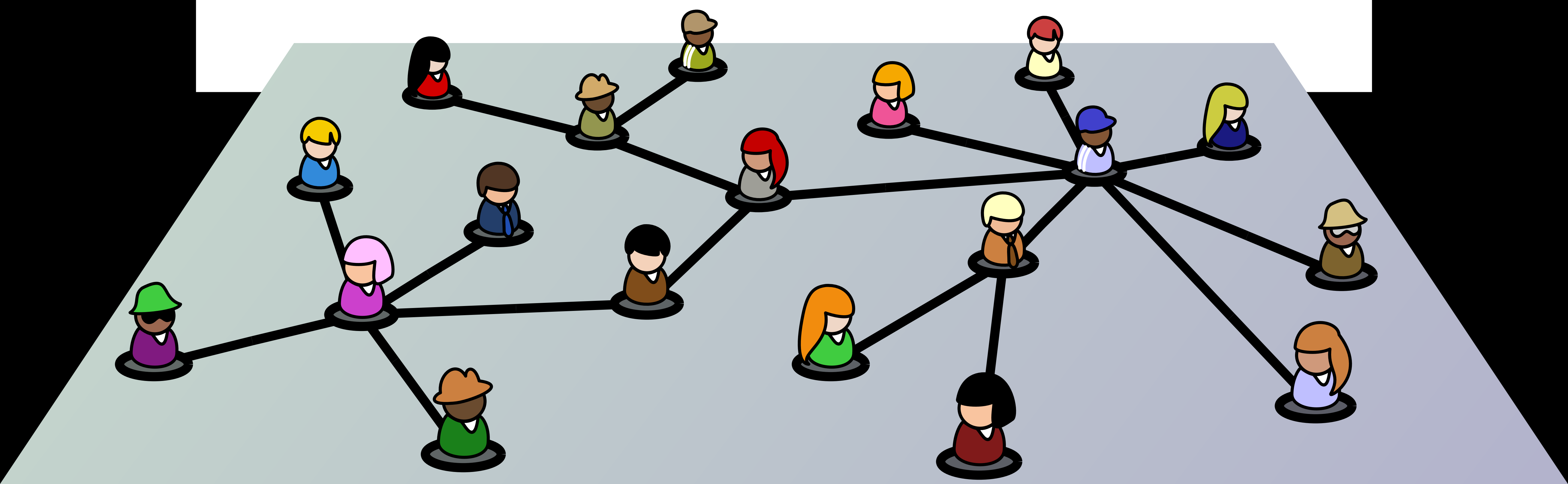

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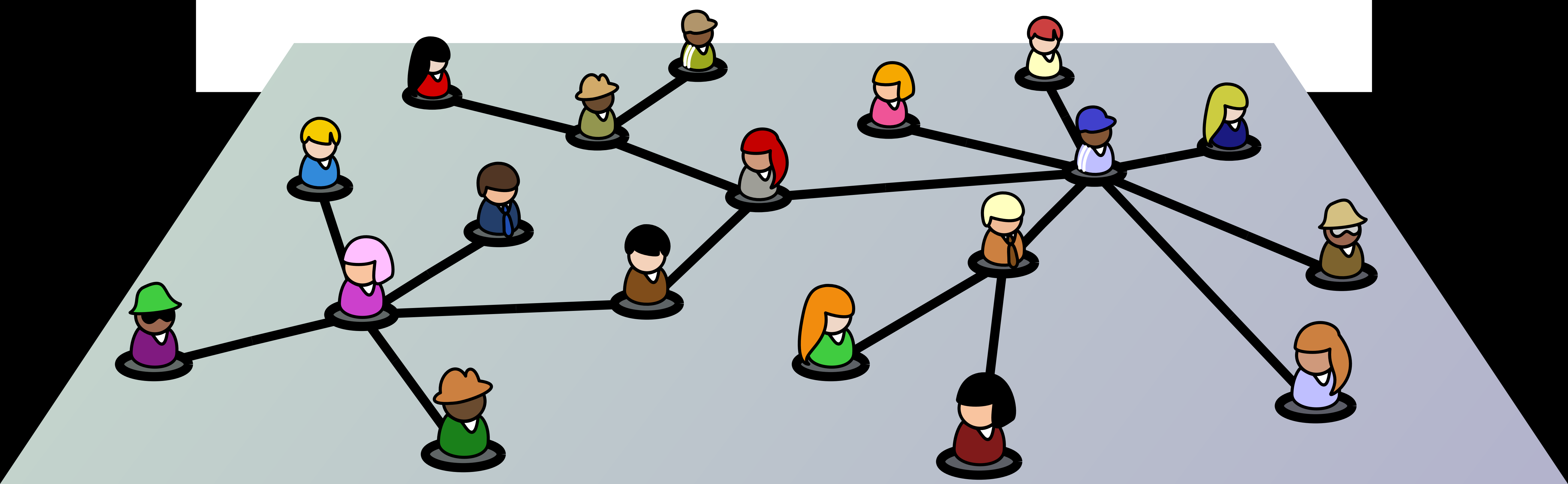

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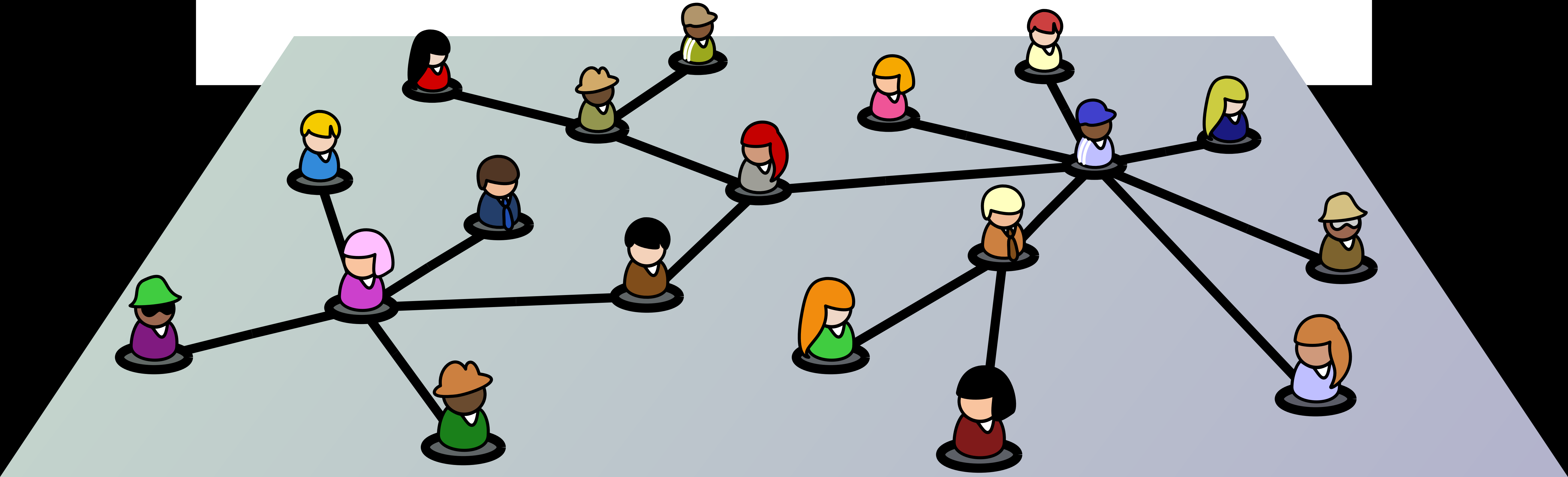

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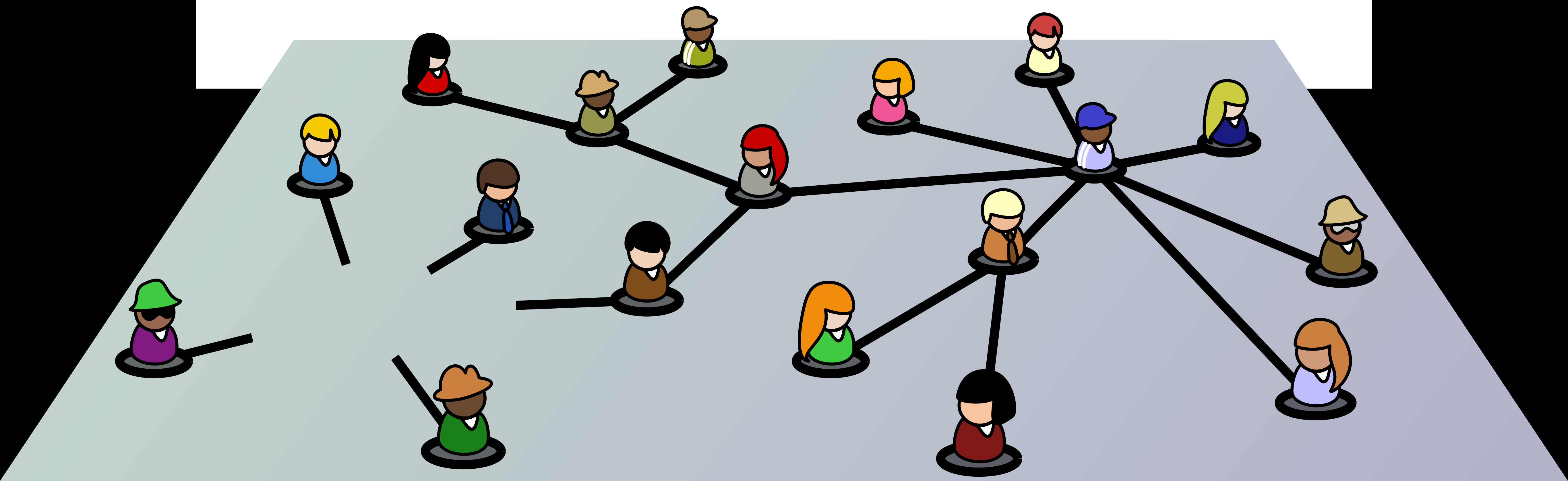

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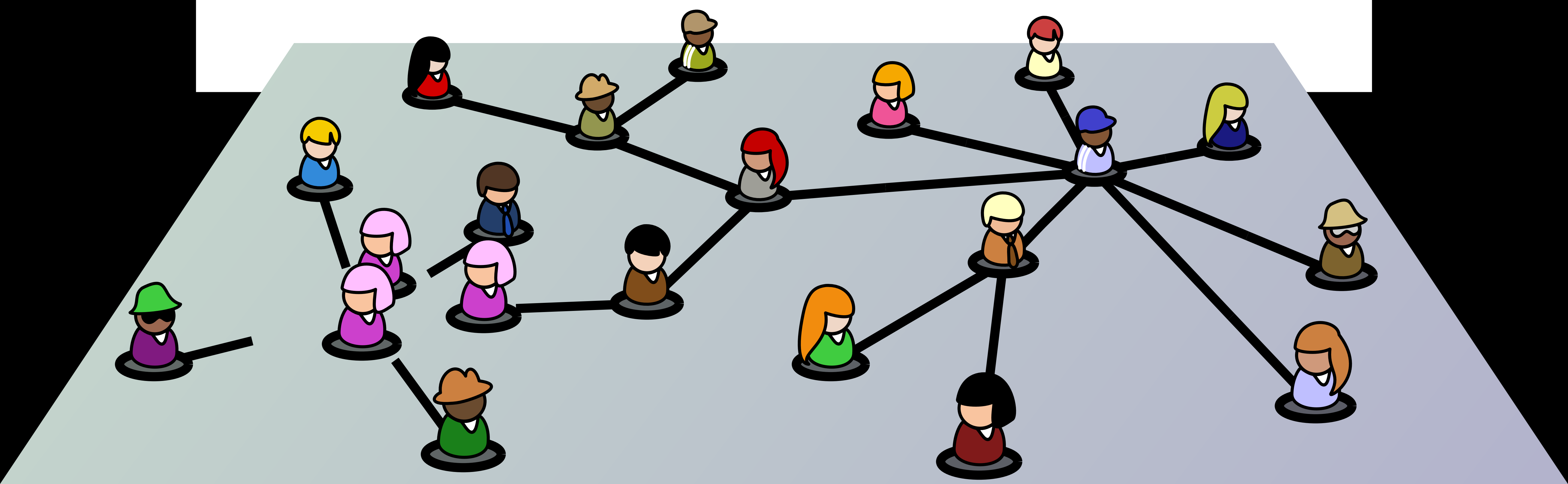

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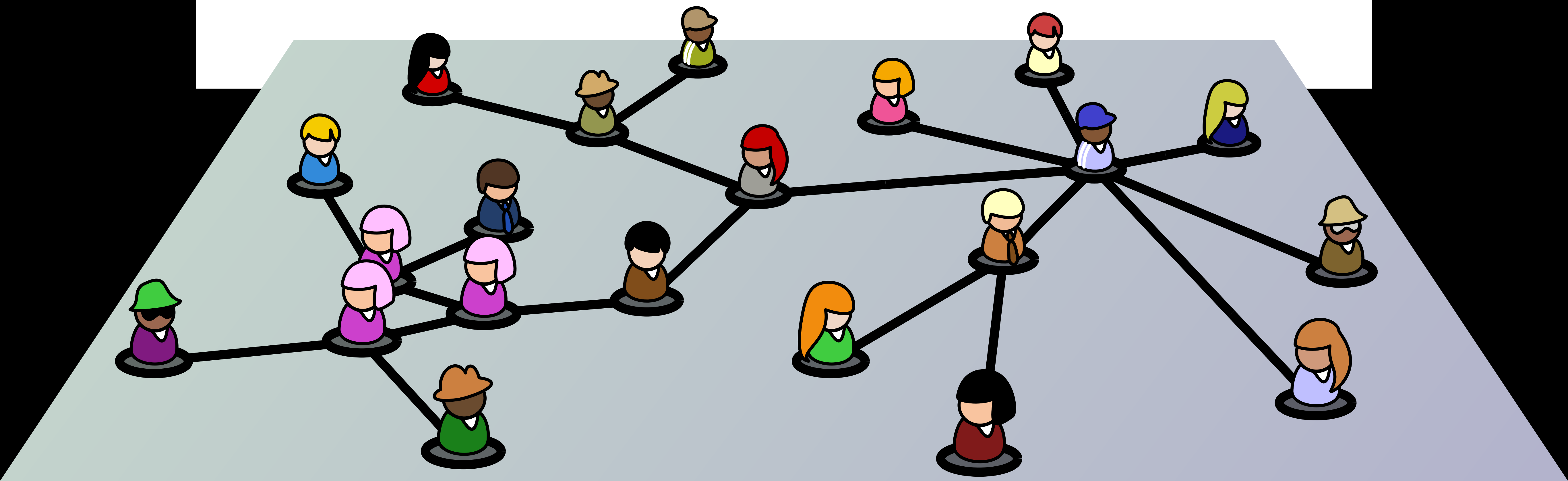

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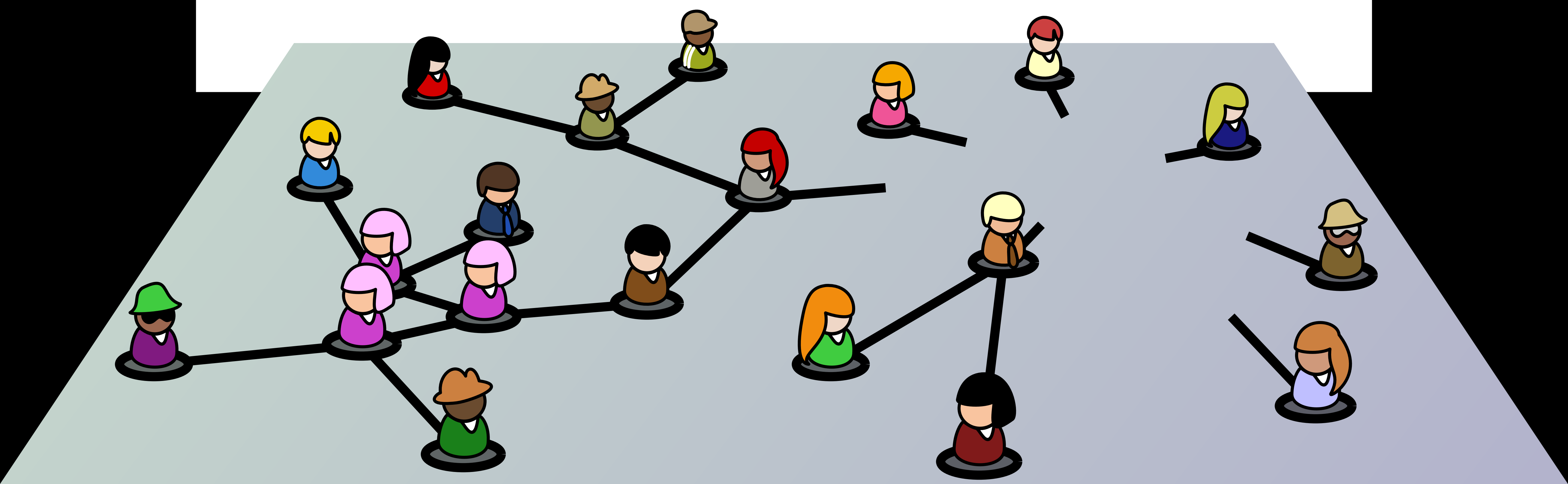

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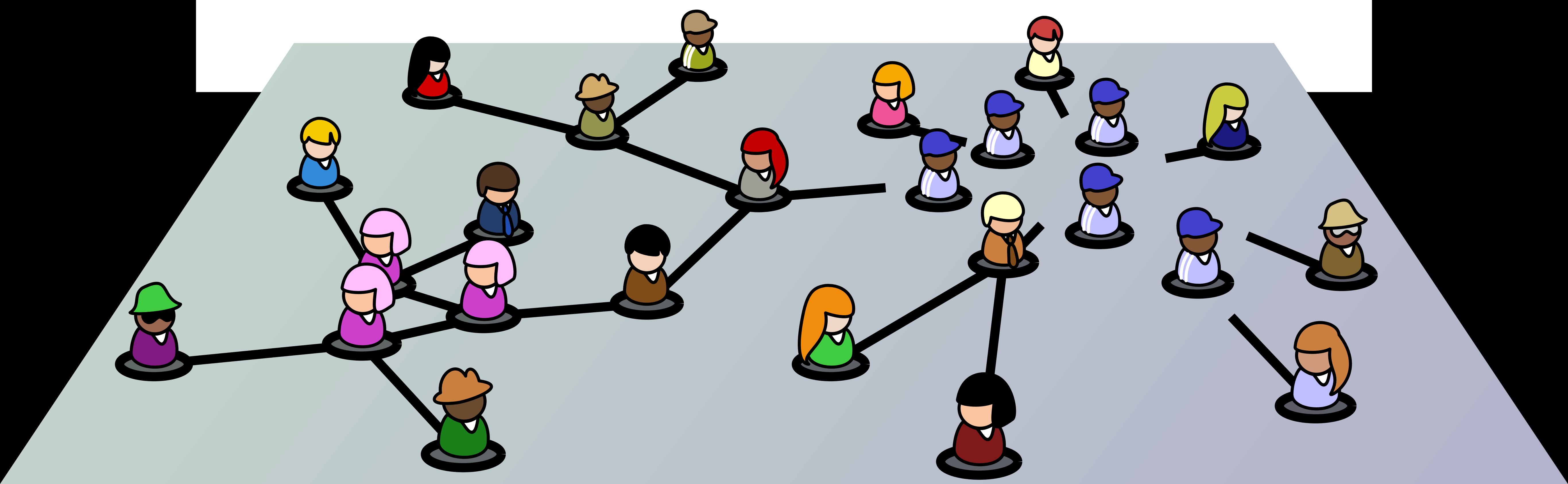

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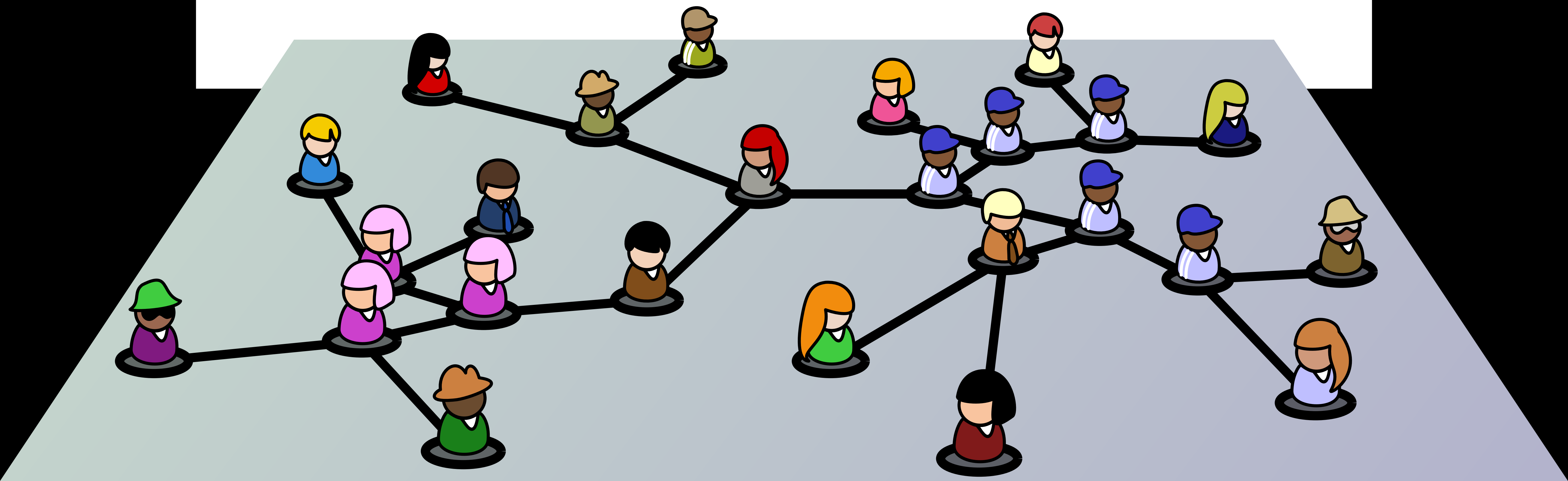

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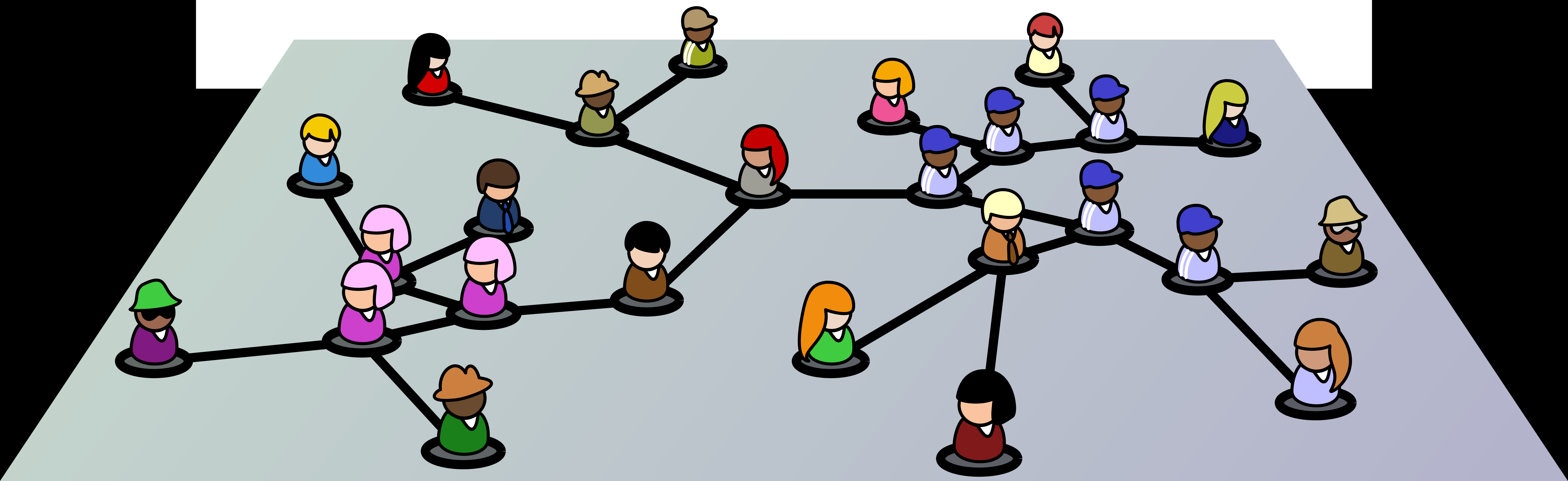

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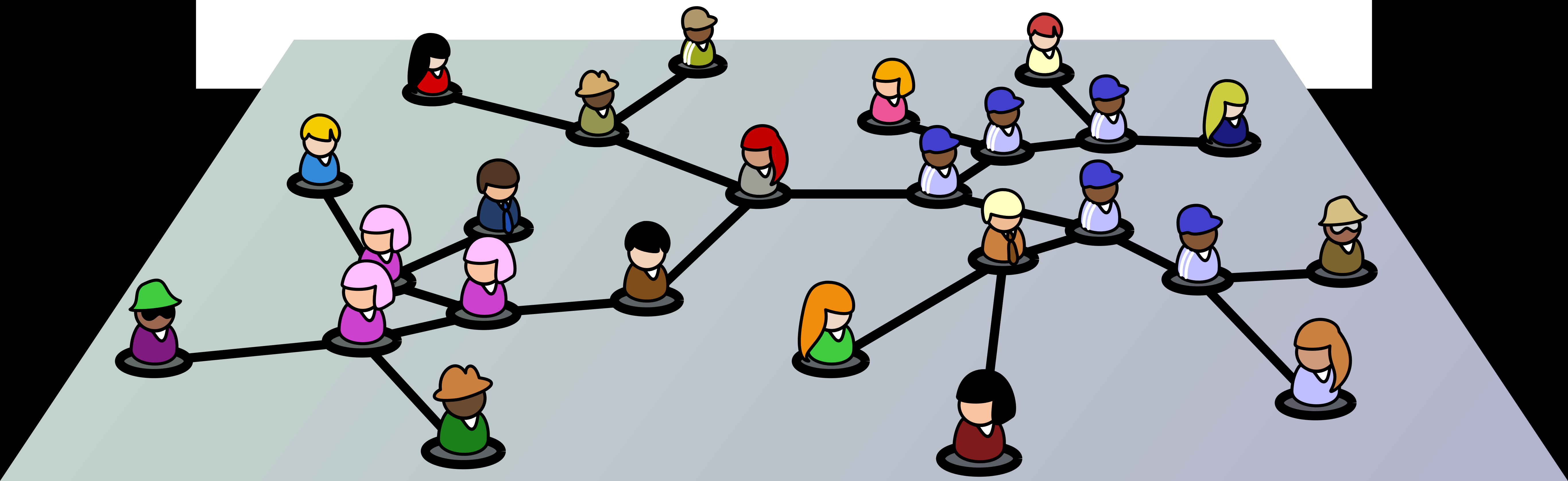

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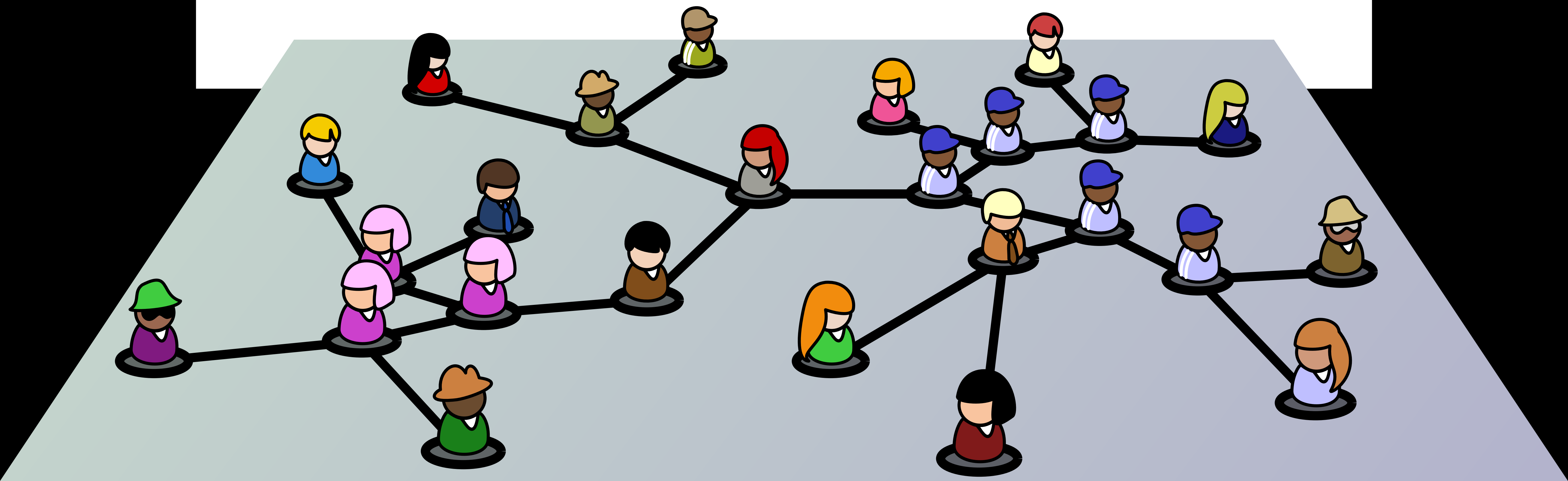

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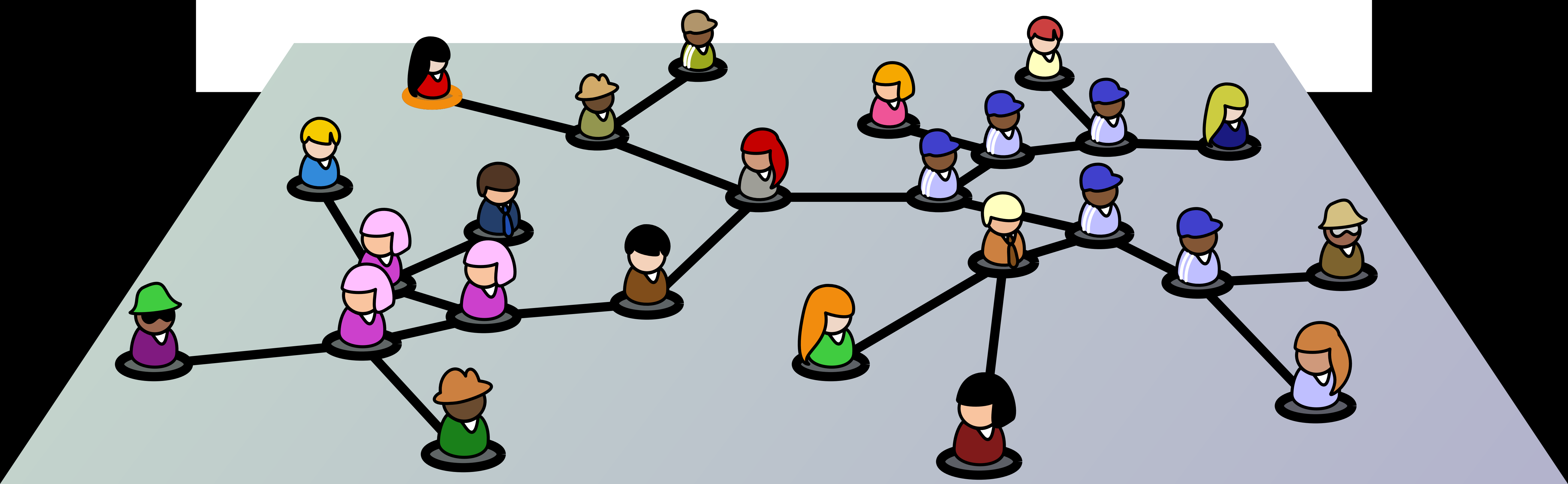

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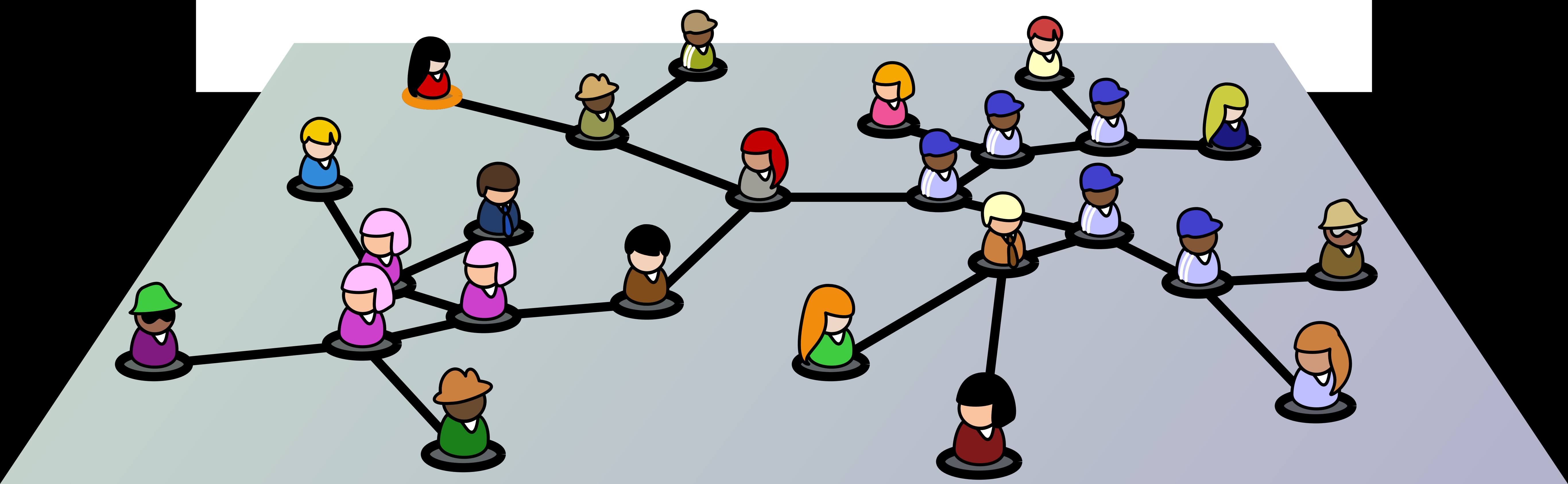

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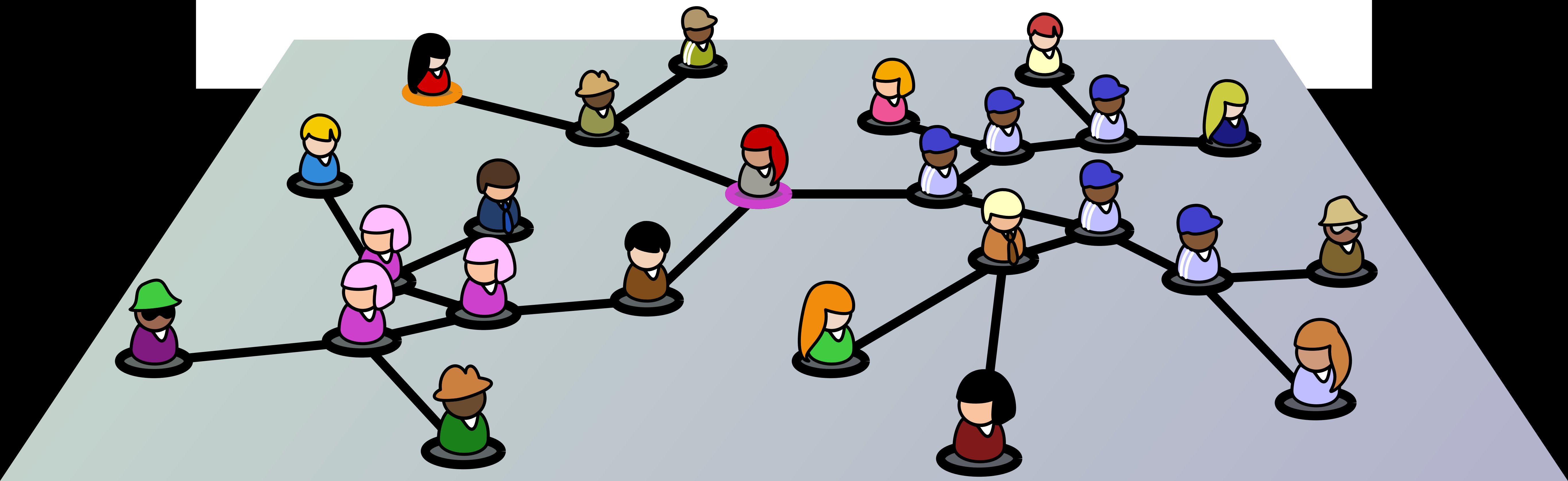

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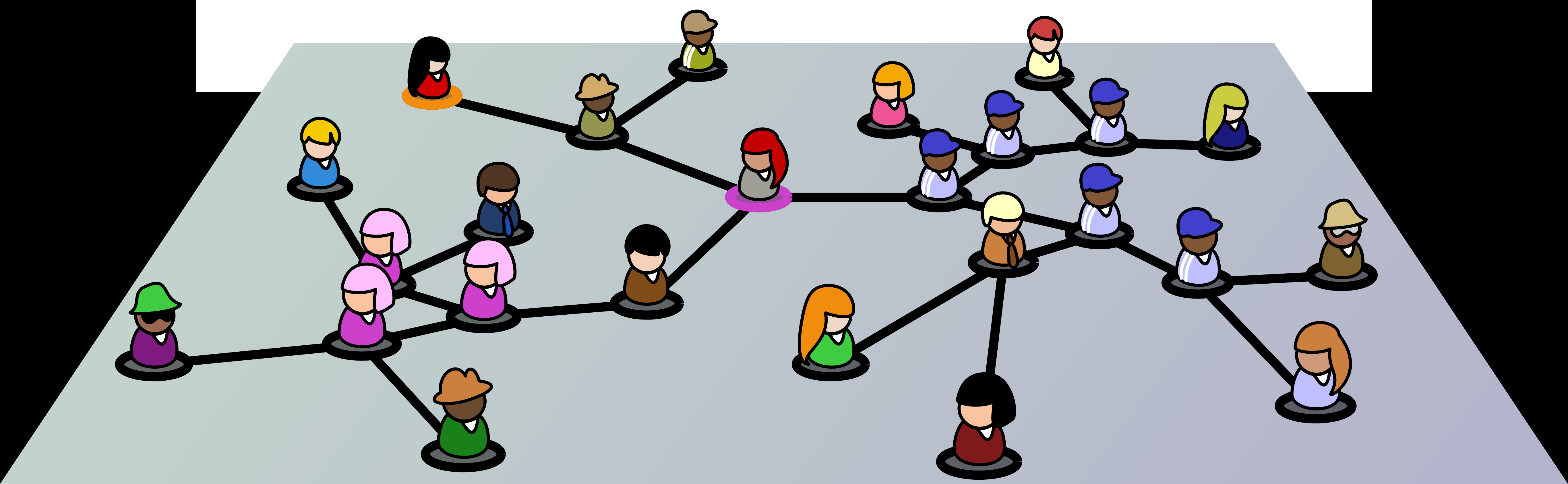

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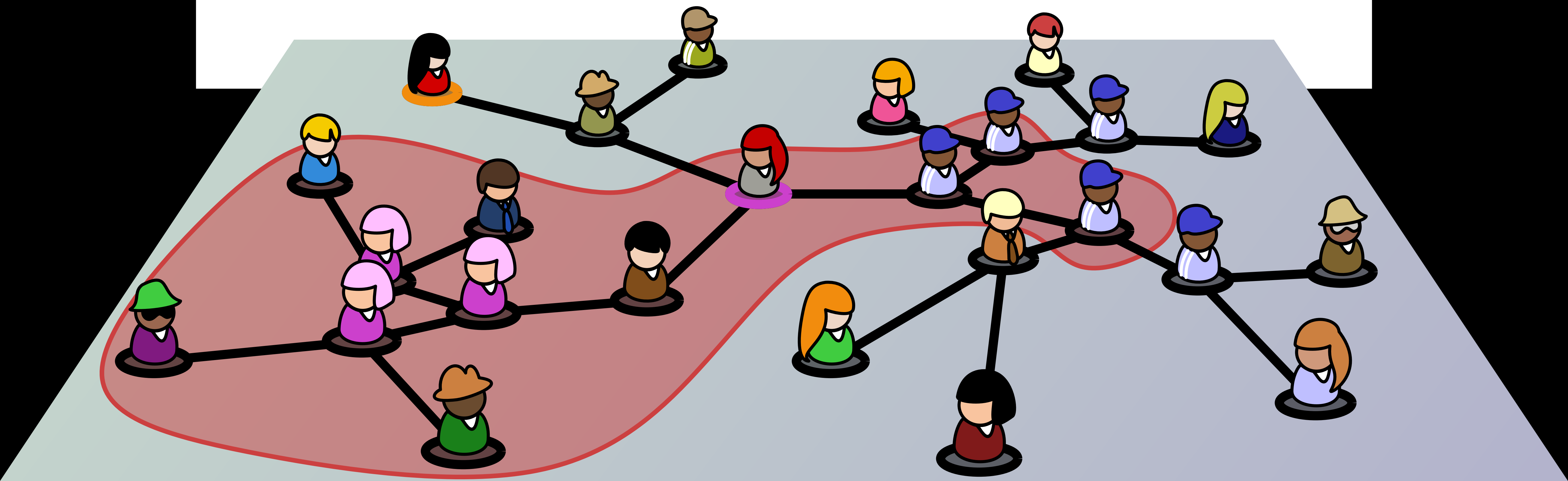

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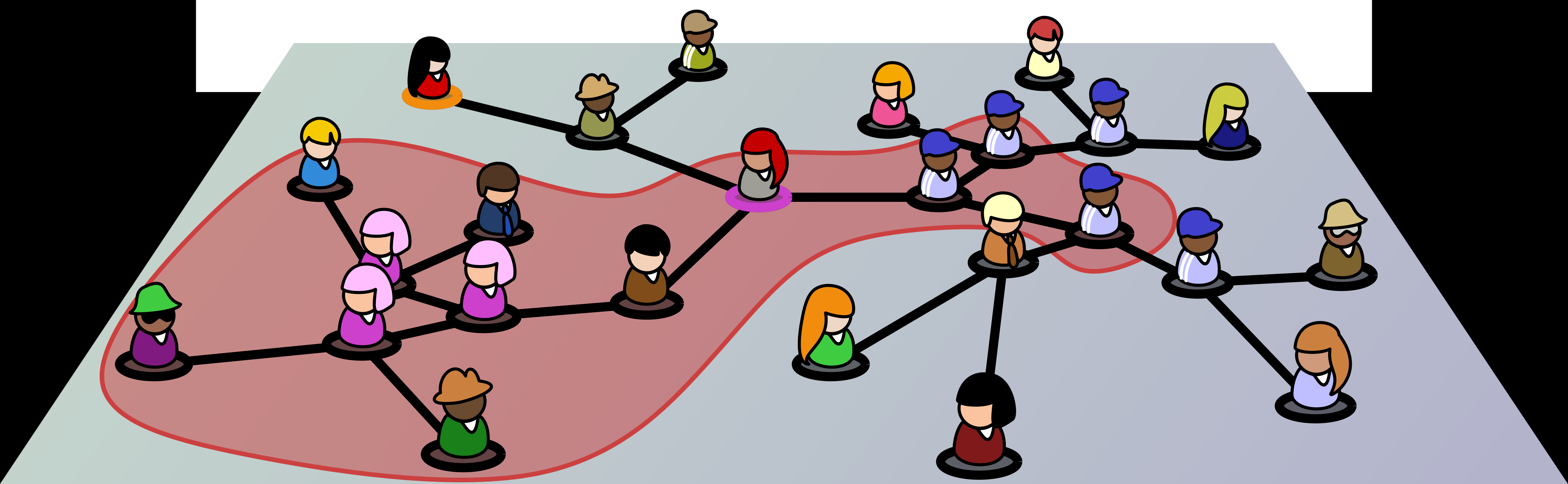

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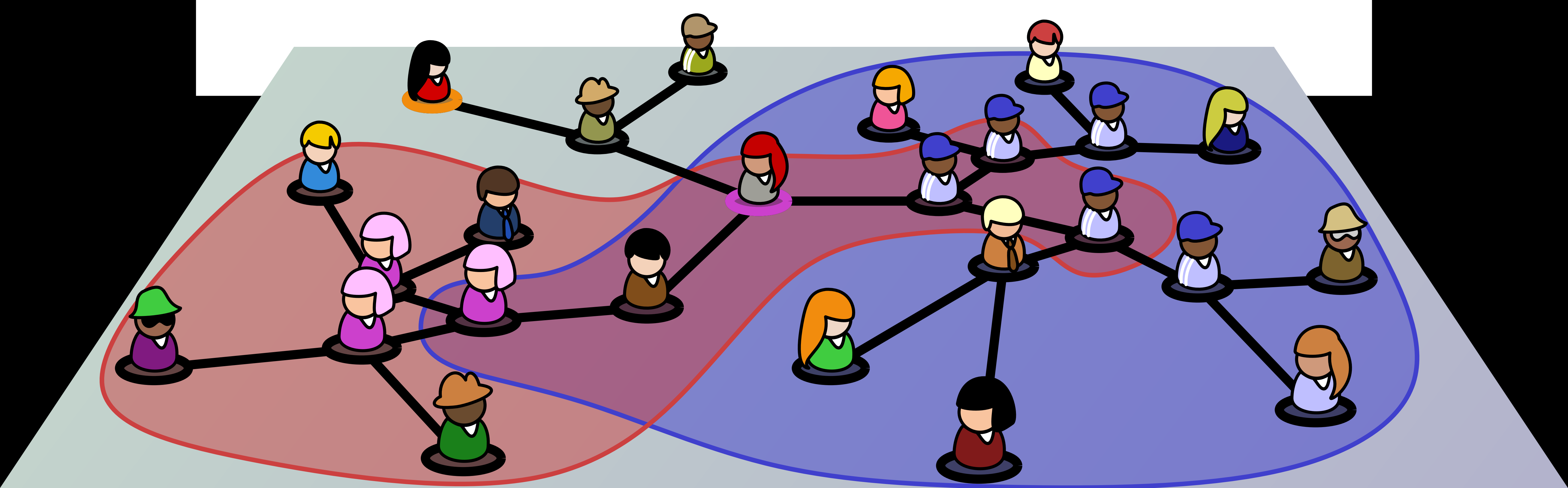

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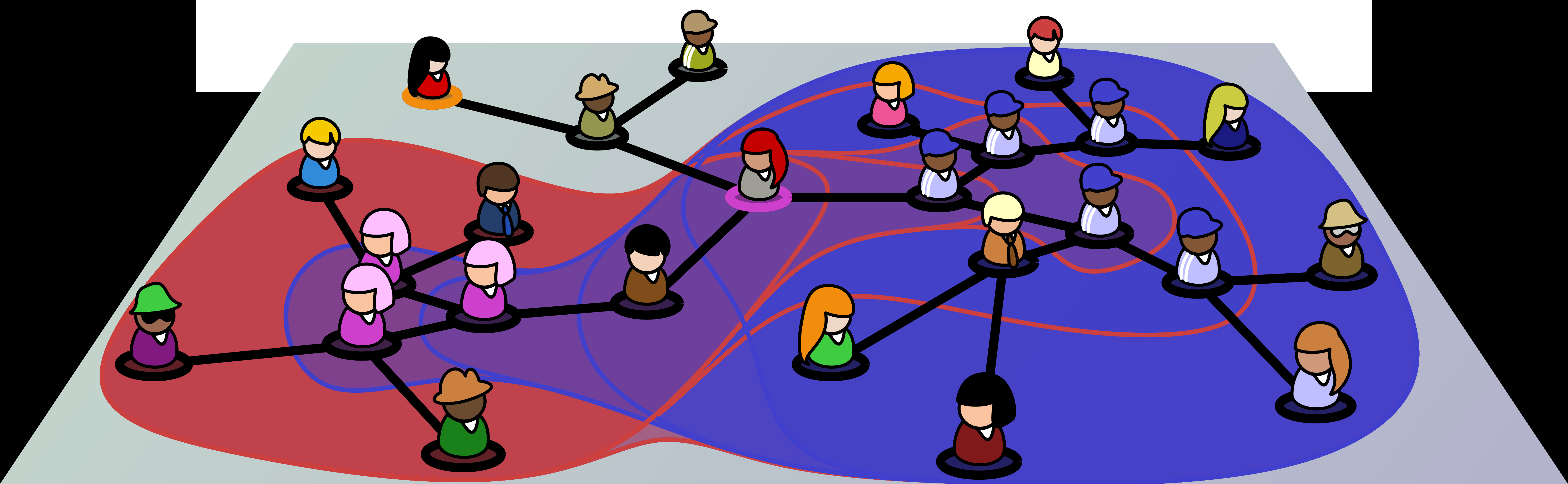

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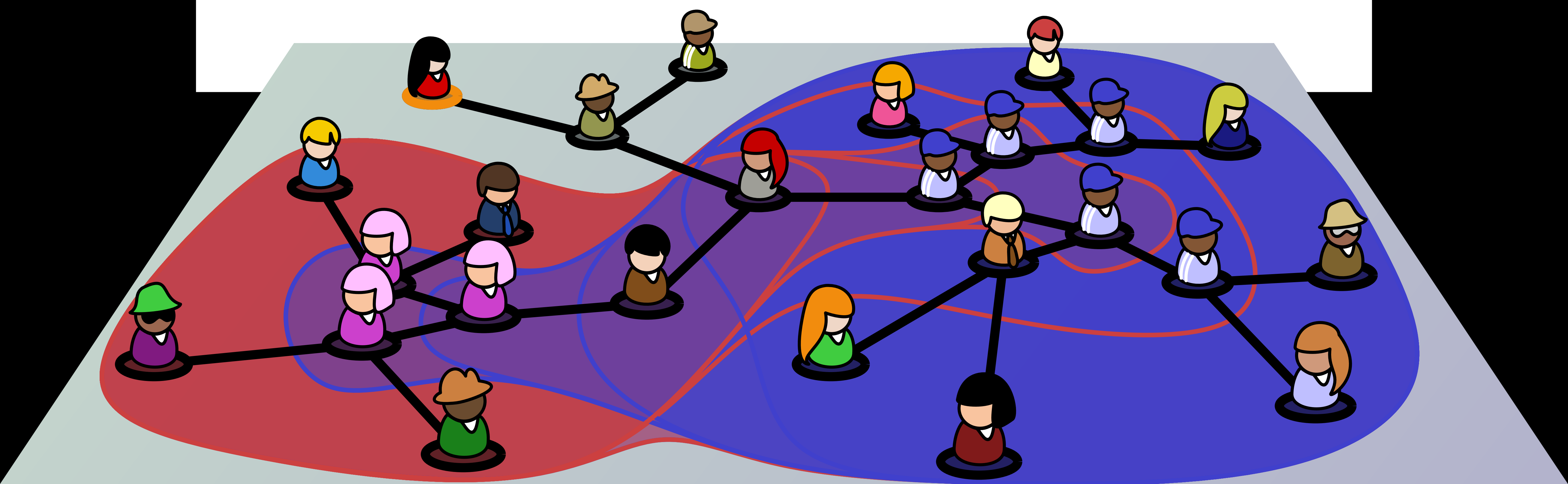

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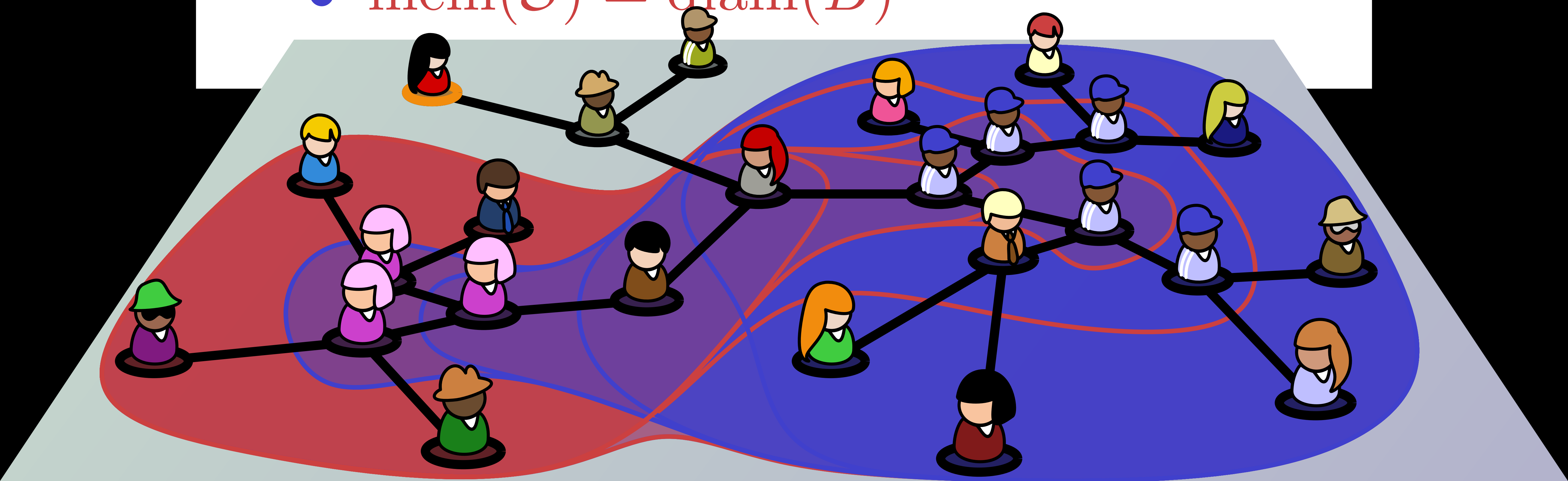

PART IV
DISCUSSION

CONCLUSION

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  - For any given graph, there exists a set of categories of low membership dimension that makes simple routing work
  - Theoretical evidence that category-based routing is a feasible explanation of Milgram's experiment

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  - Can the routing strategy be made stronger in a fair way?

THANK YOU!