Bias-Adjusted Maximum Likelihood Estimation
Improving Estimation for Exponential-Family Random Graph Models (ERGMs)

Ruth M Hummel
David R Hunter

Department of Statistics, Penn State University

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Motivation: Why model networks?

A statistical model for observed network data $y^{obs}$ allows us to:

- **Summarize**: Give a parsimonious quantitative summary of the data and, ideally, how precisely we know this summary
- **Predict**: Describe or simulate other networks that could have arisen from the same process
Motivation: The likelihood function and MLE

The ERG model class:

\[ P_\theta(Y = y) = \frac{\exp\{\theta^t g(y)\}}{\kappa(\theta)}, \quad \text{where} \quad \kappa(\theta) = \sum_{\text{all possible graphs } z} \exp\{\theta^t g(z)\} \]

- \( \theta \) is a parameter vector to be estimated.
- \( g(y) \) is a user-defined vector of graph statistics.
- The loglikelihood function is
  \[ \ell(\theta) = \theta^t g(y^{\text{obs}}) - \log \kappa(\theta). \]
- The MLE is the maximizer \( \hat{\theta} \) of the likelihood.
The likelihood is sometimes intractable.

For this undirected, 34-node network, computing $\ell(\theta)$ directly requires summation of

$$7,547,924,849,643,082,704,483,$$
$$109,161,976,537,781,833,842,$$
$$440,832,880,856,752,412,600,$$
$$491,248,324,784,297,704,172,$$
$$253,450,355,317,535,082,936,$$
$$750,061,527,689,799,541,169,$$
$$259,849,585,265,122,868,502,$$
$$865,392,087,298,790,653,952$$

terms.
The pseudolikelihood: A tractable alternative

- Some algebra based on the ERGM gives, for all $i \neq j$,
  \[
  \log \frac{P(Y_{ij} = 1 \mid Y_{ij}^c)}{P(Y_{ij} = 0 \mid Y_{ij}^c)} = \theta^t \left[ g(Y_{ij}^+) - g(Y_{ij}^-) \right].
  \]

- The pseudolikelihood ignores the conditioning, assuming instead
  \[
  \log \frac{P(Y_{ij} = 1)}{P(Y_{ij} = 0)} = \theta^t \left[ g(Y_{ij}^+) - g(Y_{ij}^-) \right] \equiv \theta^t \delta(Y)_{ij}
  \]
  independently for all $i \neq j$.

- Thus, the pseudolikelihood equals
  \[
  \prod_{i \neq j} \frac{\exp \left\{ \theta^t \delta(y_{ij}^{\text{obs}}) \right\} y_{ij}^{\text{obs}}}{1 + \exp \left\{ \theta^t \delta(y_{ij}^{\text{obs}}) \right\}}
  \]
Van Duijn, Gile, and Handcock (2009, *Social Networks*) compare MLE to MPLE.

- They cite a small but compelling set of explorations of the MPLE, suggesting that there may be large differences between the MPLE and the approximate MLE, sometimes even in cases where the dependence is not thought to be a concern.
- They explore the bias in the MLE and MPLE compared to the “truth”
- They introduce a bias-corrected version of the MPLE (the “MBLE”).
- A similar bias-correction is possible for the MLE, though it is a bit less straightforward.
The bias-correction we employ (which might be better described as a preemptive bias-*mitigation*, rather than correction) follows from Firth (1993). The idea is to maximize a penalized likelihood which induces a bias in the score function in order to reverse the some of the anticipated bias in the maximizer. The penalized likelihood is:

\[
\ell_{bc}(\theta) = \ell(\theta) + \frac{1}{2} \log |I(\theta)|
\]

The resulting maximizer is also the Bayesian maximum posterior estimator based on assigning a Jeffreys prior to the parameter.
The intuition behind this modification for an exponential family model is the following: Since the score function, $U(\eta)$, can be written

$$U(\eta) = \ell'(\eta) = g(Y) - \kappa'(\eta),$$

it is clear that the shape of $U(\eta)$ is not affected by the sufficient statistic, $g(Y)$. For this reason, any anticipated bias in the MLE can be offset by shifting the score function by the amount $bias \ast \nabla U$. (Here $\nabla U = -i(\eta)$.) This adjustment is illustrated in the following figure, taken from Firth (1993):

**Figure**: Modification of the unbiased score function
Evidence of bias in MLE (and MPLE) compared to “truth”

Taken from van Duijn, et al. (2009), these boxplots show the bias of the MLE for selected parameters in two networks (“original” and “transitivity”) for the canonical parameter space. (The true parameter is shown as a horizontal line.) Note that the bias is greatest in the MLE.

Fig. 1. Boxplots of the distribution of the MLE, the MPLE and the MBLE of the geometrically weighted edgewise shared partner statistic (GWESP), differential activity by practice statistic (Nodal), and homophily on practice statistic (Homophily) under the natural and mean value parameterization for 1000 samples of the original Lazega network and 1000 samples of the Lazega network with increased transitivity. (a) Natural parameterization, GWESP; (b) mean-value parameterization, GWESP; (c) natural parameterization, node practice; (d) mean-value parameterization, node practice; (e) natural parameterization, homoph practice; (f) mean value parameterization, homoph practice.

Fig. 1 (b), (d) and (f) demonstrate that with negligible bias and substantially smaller variance, the MLE clearly out-performs the pseudo-likelihood methods in the mean-value parameter space. There is a pronounced left skew to the mean-value parameter estimates for the pseudo-likelihood methods. The bivariate scatter plots in Fig. 2 suggest that it is this set of samples with very low mean-value parameter estimates that account for much of the bias, and therefore efficiency loss of the pseudo-likelihood methods. The MBLE performs better than the MPLE because its estimates are less skewed than those of the MPLE.
Here we see that there is no bias of the MLE for selected parameters in two networks ("original" and "transitivity") for the mean value parameter space. (This is by definition, since the mean-value MLE is the observed statistic.)
In order to compare our present extended results to the results found for just the MBLE and the ordinary MPLE and MLE in the van Duijn, et al. paper, we duplicate their results on the corporate lawyer partnerships data and include the analysis for the bias-corrected MLE (pMLE).
Lazega collaboration network

The Lazega collaboration data are collaborations in the late 1980’s between 36 New England lawyers determined by their responses to the question “With which members of your firm have you spent time together on at least one case, have you been assigned to the same case, have they read or used your work product or have you have read or used their work product?”

Additional member attributes collected include the attorneys’ gender, age, status (36 are partners; 35 are associates), seniority, years with the firm, practice (litigation or corporate), office location (Boston, Hartford, or Providence), and law school attended (Yale or Harvard, University of Connecticut, or any other).
Following van Duijn, et al., we simulate networks based on a "truth" for the following model:

<table>
<thead>
<tr>
<th>Model terms</th>
<th>&quot;True&quot; parameter value</th>
</tr>
</thead>
<tbody>
<tr>
<td>edges</td>
<td>-6.506</td>
</tr>
<tr>
<td>GWESP</td>
<td>0.897</td>
</tr>
<tr>
<td>seniority (nodal covariate)</td>
<td>0.853</td>
</tr>
<tr>
<td>practice (nodal covariate)</td>
<td>0.410</td>
</tr>
<tr>
<td>practice (homophily effect)</td>
<td>0.759</td>
</tr>
<tr>
<td>gender (homophily effect)</td>
<td>0.702</td>
</tr>
<tr>
<td>office (homophily effect)</td>
<td>1.145</td>
</tr>
</tbody>
</table>
Preliminary results:

Results based on very few simulations show no improvement in the MLE yet...

**Figure:** Distribution of the GWESP and Nodal Practice canonical parameter; true parameter shown as horizontal line.
Preliminary results:

Here you can see that the number of sub-simulations for calculating the mean value parameter is clearly not sufficient, as the mean for the uncorrected MLE should be unbiased...

Figure: Distribution of the GWESP and Nodal Practice mean value parameter; true parameter shown as horizontal line.
Current extensions:

- increasing the simulations for the current network
- applying the same to the “increased transitivity” version of the collaboration network as used in van Duijn, et al.
- applying the same to a larger biological network
- applying the same to a friendship network
Consider the idea of MCMC MLE:

- Suppose we fix $\eta_0$. A bit of algebra shows that

$$- \log E_{\eta_0} \left[ \exp \left\{ (\eta - \eta_0)^t g(Y) \right\} \right] = \ell(\eta) - \ell(\eta_0).$$  \hspace{1cm} (1)

- The Law of Large Numbers suggests obtaining a sample of $Y$ from the model using $\theta_0$ as the parameter, then approximating the expectation by a sample mean.

- Q: How do we sample from $g(Y)$ using $\theta_0$ as the parameter?
  A: Run MCMC infinitely long.
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- But what if we only run MCMC for a single step (starting at $y_{obs}$), for a randomly chosen $Y_{ij}$?

- For this $Y_{ij}$, we’re sampling from the conditional distribution given $(y_{obs})_{ij}$. 
To summarize:

- Running an infinitely long Markov chain leads to the loglikelihood.
- Running a 1-step Markov chain leads to the pseudolikelihood.

Thus, if we alternately sample and then optimize the resulting “likelihood-like” function, we can view MLE and MPLE as two ends of a spectrum, the “contrastive divergence” spectrum. (MLE is CD-$\infty$ and MPLE is CD-1.)
A few words about Contrastive Divergence (CD)

Considering CD-1...

Q: Is it better to

1. Repeatedly pick $i \neq j$ at random, or
2. Cycle through all possible $i \neq j$ in some systematic fashion?

A: The latter. The reason boils down to the following well-known identity for any two random variables $Y$ and $Z$:

$$\text{Var}(Y) = \text{Var}[E(Y | Z)] + E[\text{Var}(Y | Z)].$$

Here, “$Y$” is the likelihood-like quantity based on the randomly sampled networks and “$Z$” is the selected pair $i \neq j$. 