Near-optimal Fixed-parameter Tractability of the Bron–Kerbosch Algorithm for Maximal Cliques

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Joint work with David Eppstein and Maarten Löffler

What is a Maximal Clique?

A clique that cannot be made bigger by adding more vertices

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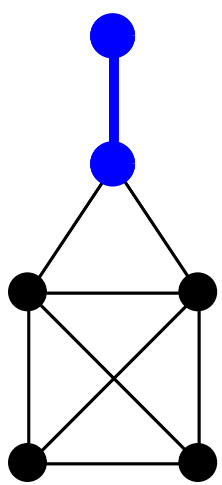
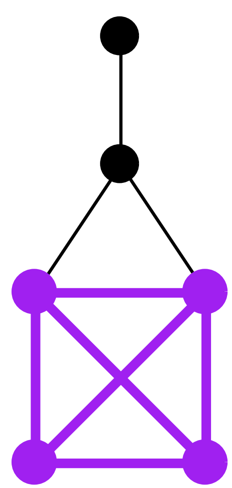
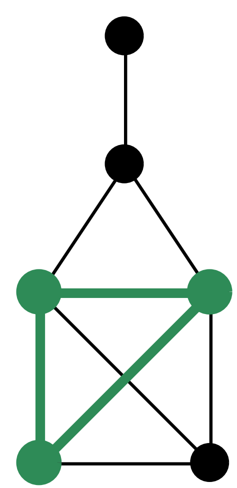
Maximal



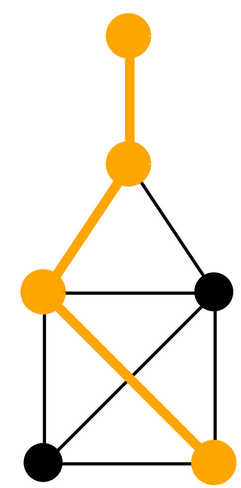
Maximal, Maximum



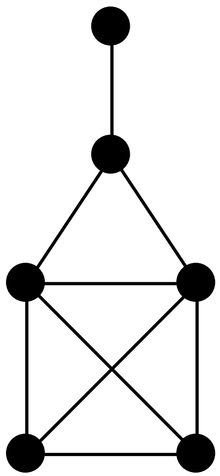
Not Maximal



Not Clique



Goal: Design an algorithm to list all maximal cliques



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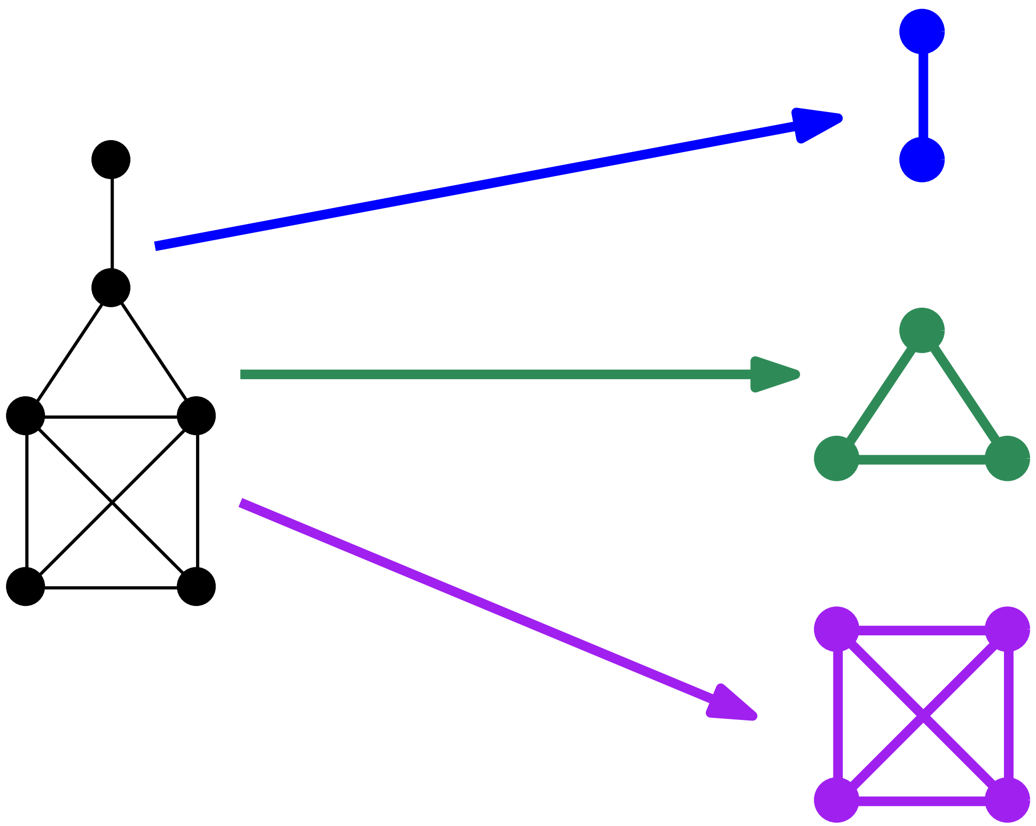

Motivation

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Features in ERGM

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Detect structural motifs from similarities between proteins

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Detect structural motifs from similarities between proteins

Determine the docking regions between biomolecules

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Determine the docking regions between biomolecules

Document clustering for information retrieval

There may be many maximal cliques.

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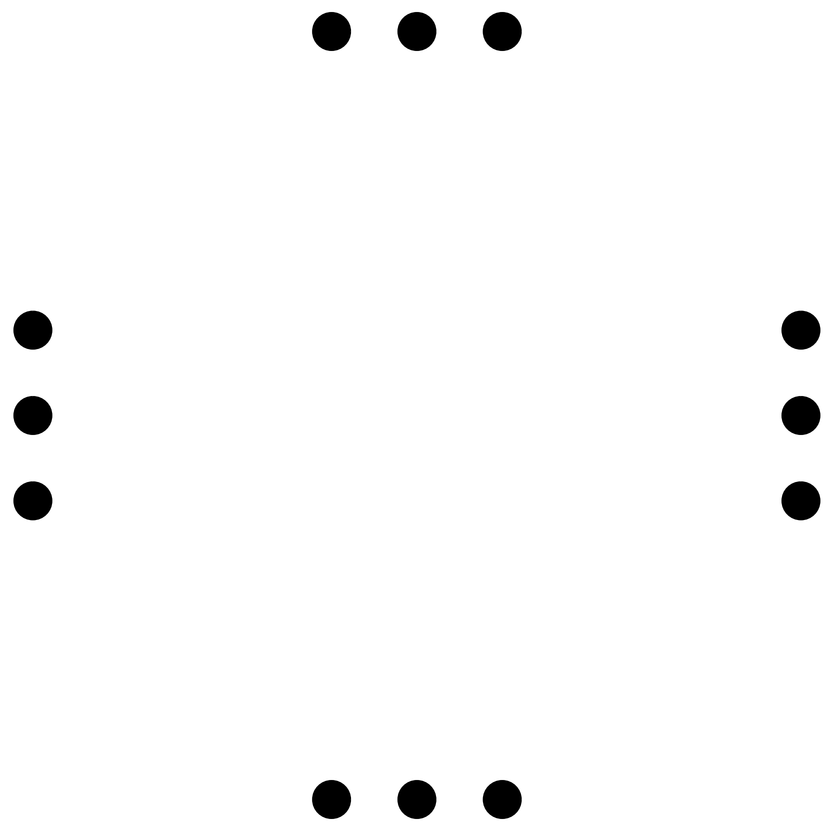

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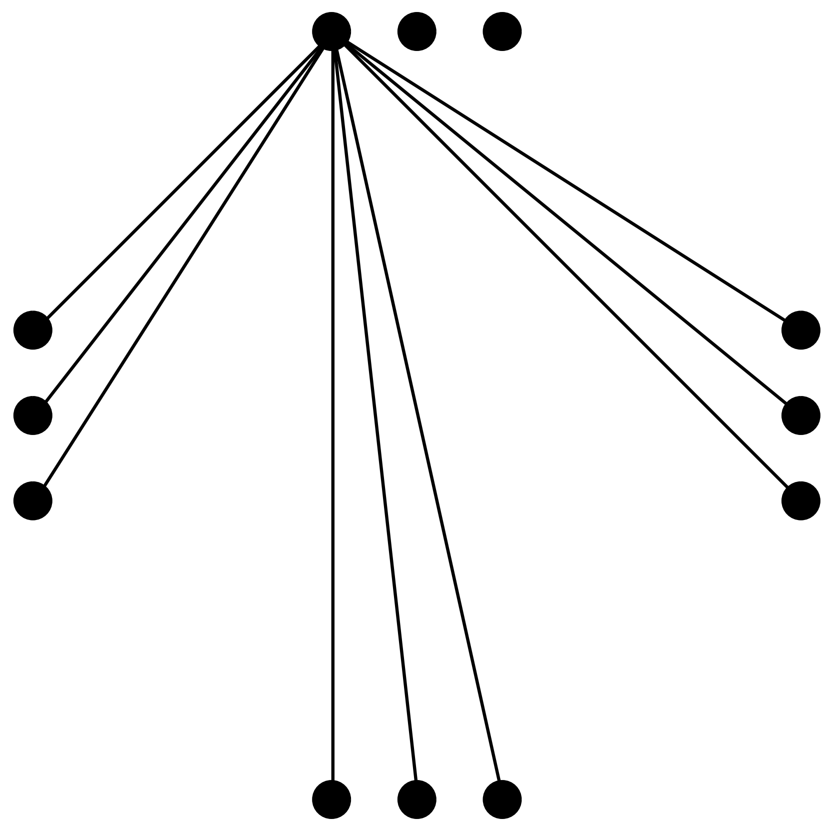

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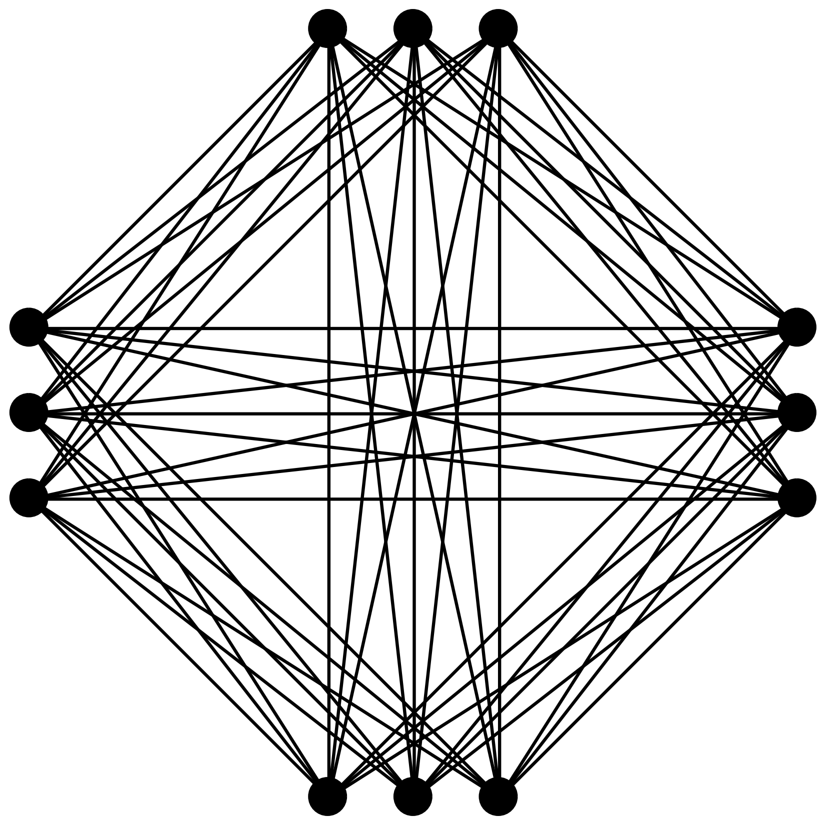

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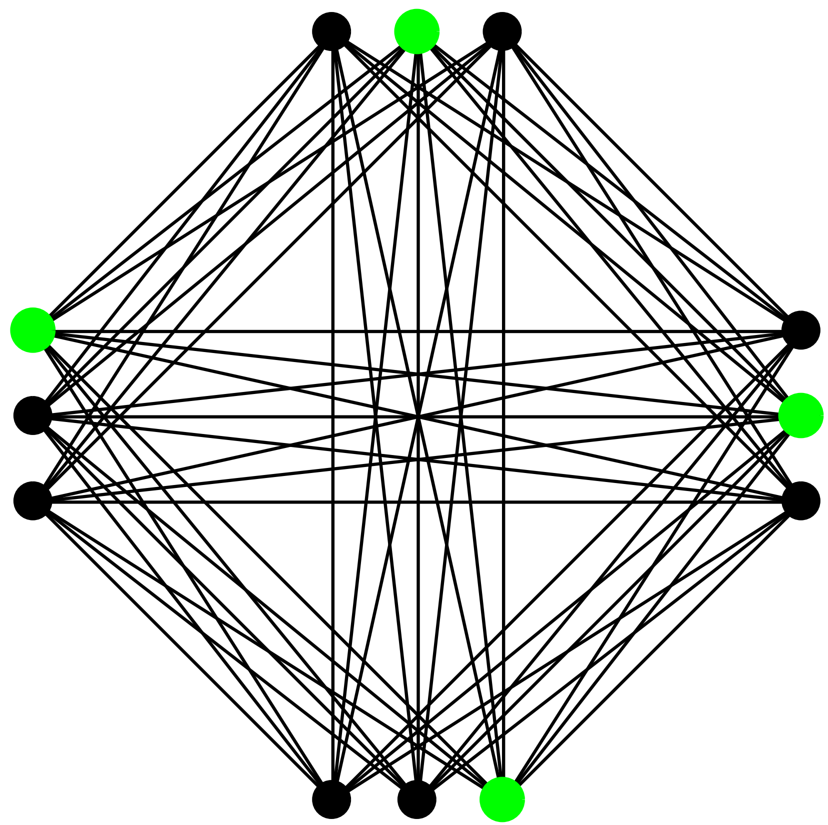

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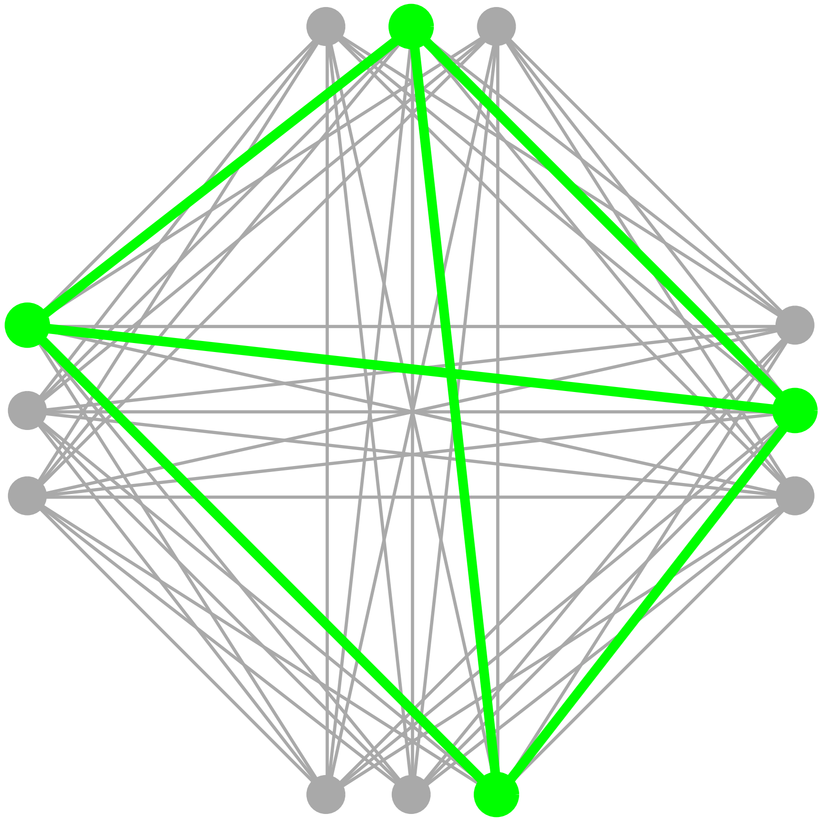

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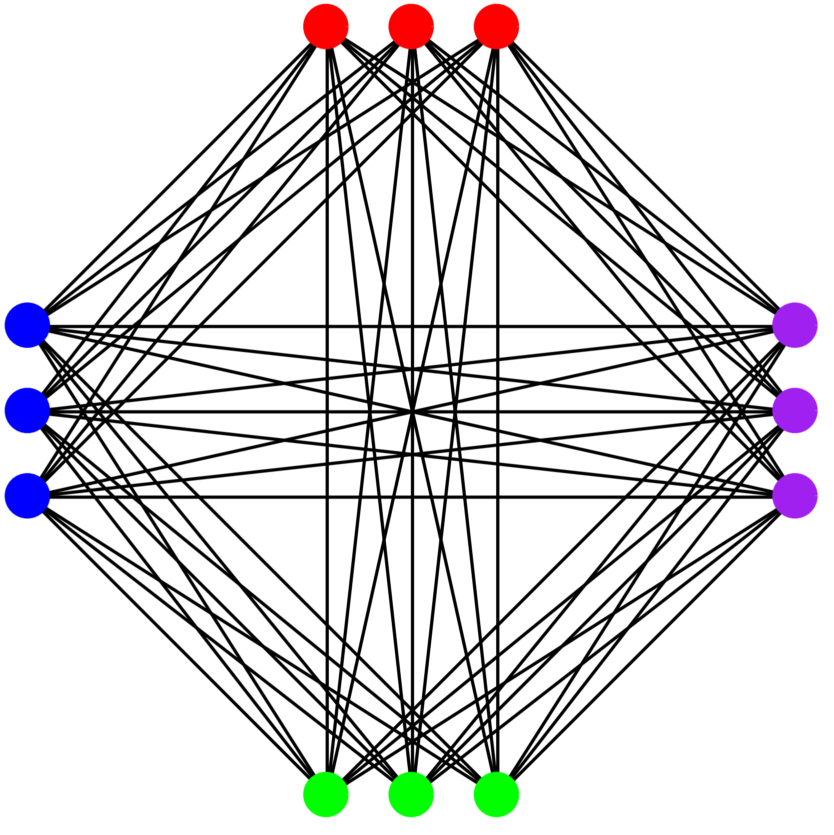

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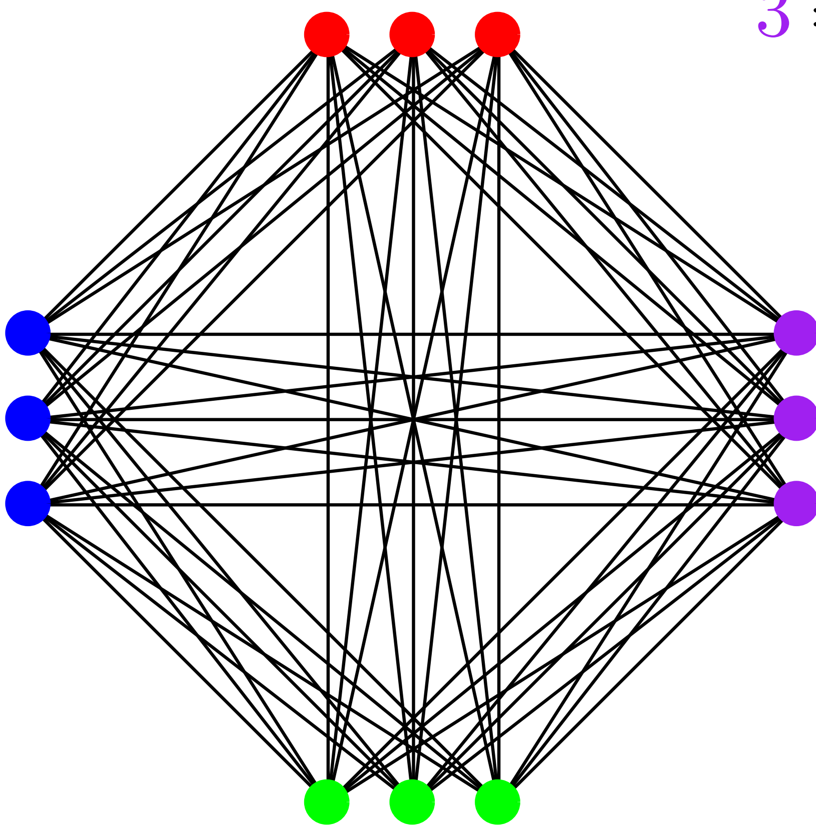
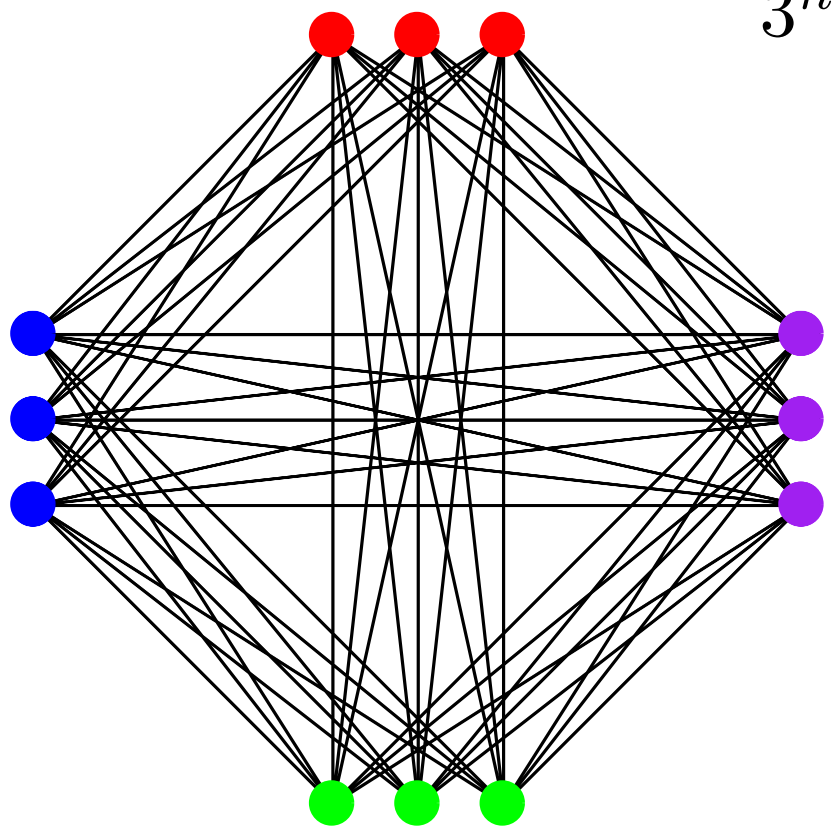


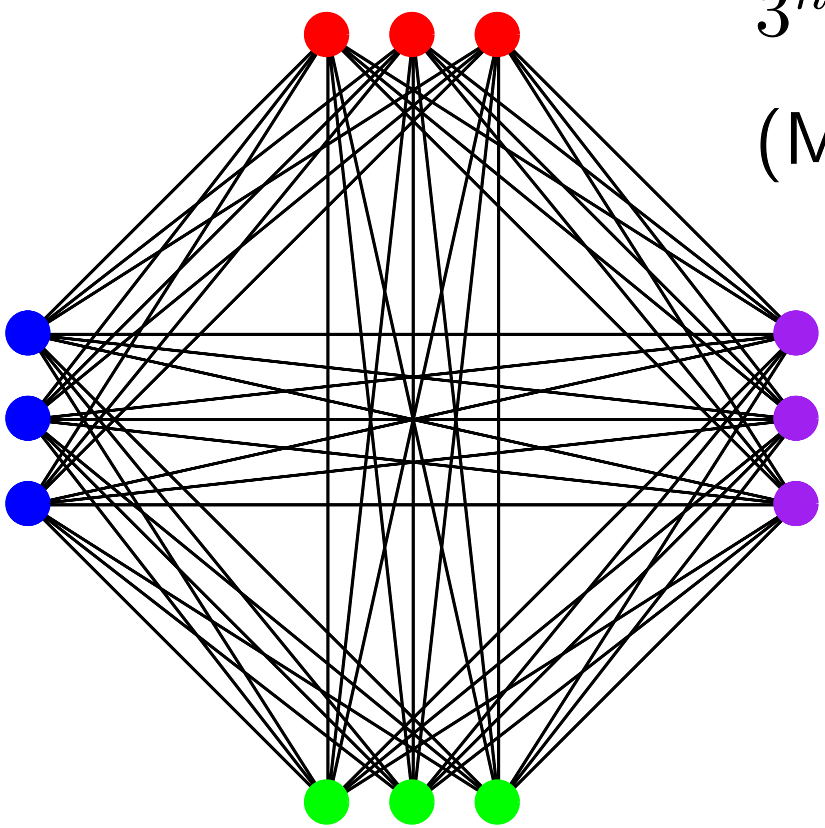
$3 * 3 * 3 * 3$

There may be many maximal cliques.



$3^{n/3}$ maximal cliques

There may be many maximal cliques.



$3^{n/3}$ maximal cliques

(Moon–Moser bound)

Maximal Clique Listing Algorithms

Author	Year	Running Time
Bron and Kerbosch	1973	???
Tsukiyama et al.	1977	$O(nm\mu)$
Chiba and Nishizeki	1985	$O(\alpha m\mu)$
Makino and Uno	2004	$O(\Delta^4\mu)$

n = number of vertices
m = number of edges
μ = number of maximal cliques
α = arboricity
Δ = maximum degree of the graph

Tomita et al. (2006)

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Worst-case optimal running time $O(3^{n/3})$

Tomita et al. (2006)

Worst-case optimal running time $O(3^{n/3})$

Computational experiments:

AMC [14]	AMC* [14]	CLIQUES
261.27	**9.51**	10.49
952.25	49.45	**10.20**
3,601.09	130.76	**9.90**
14,448.21	431.20	**10.95**
35,866.69	530.53	**12.97**
$>$ 24 h	1,066.62	**16.85**
$>$ 24 h	4,350.94	**33.75**
$>$ 24 h	15,655.05	**65.06**
$>$ 24 h	$>$ 24 h	**293.97**

The Bron–Kerbosch Algorithm

Easy to understand

Easy to implement

There are many heuristics, which make it faster

Its variations work well in practice.

Confirmed through computational experiments
Johnston (1976), Koch (2001), Baum (2003)

One variation is worst-case optimal ($O(3^{n/3})$ time)
Tomita et al. (2006)

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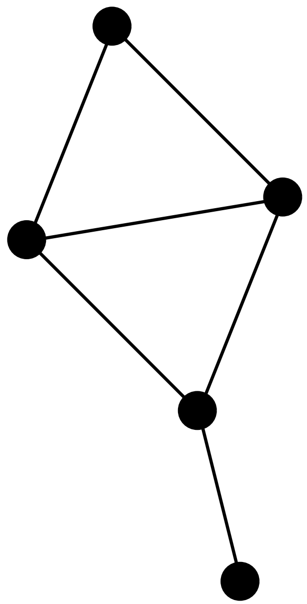

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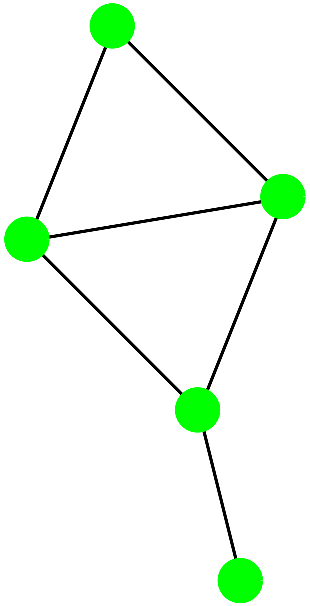

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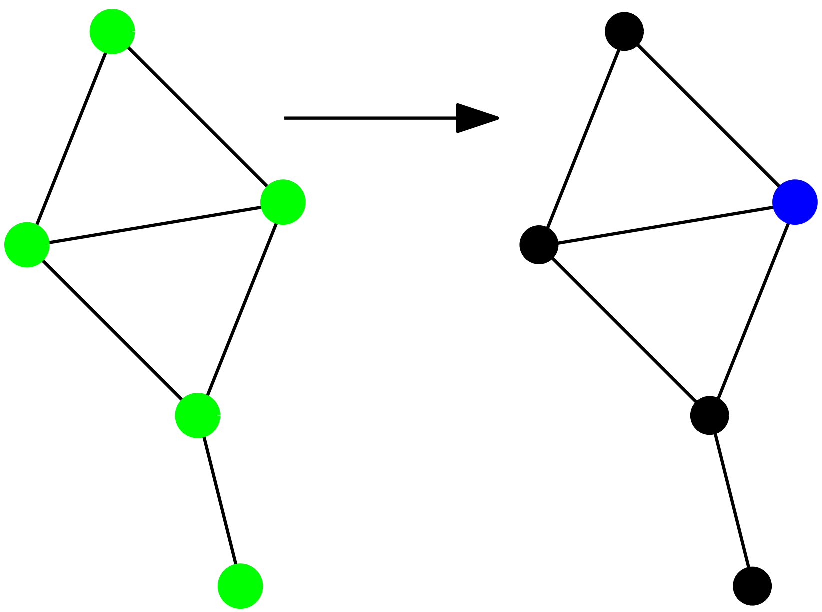

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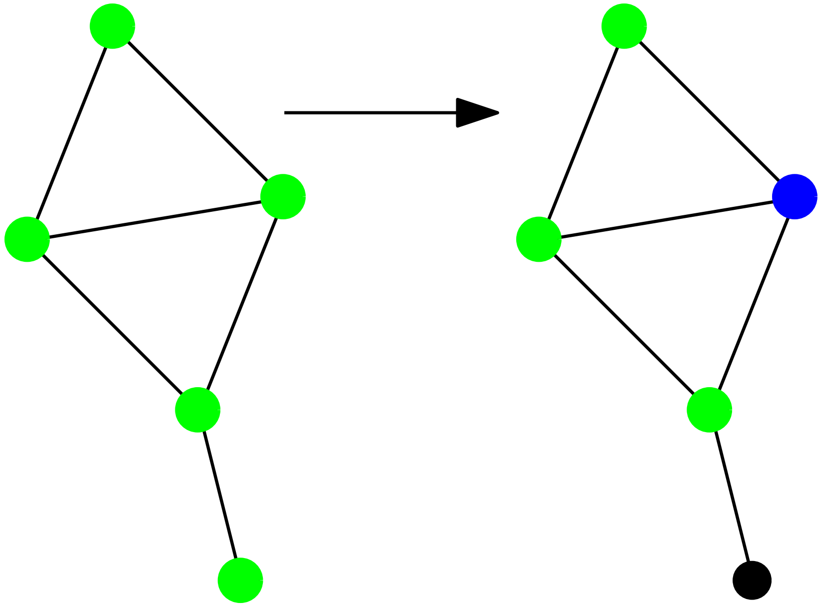

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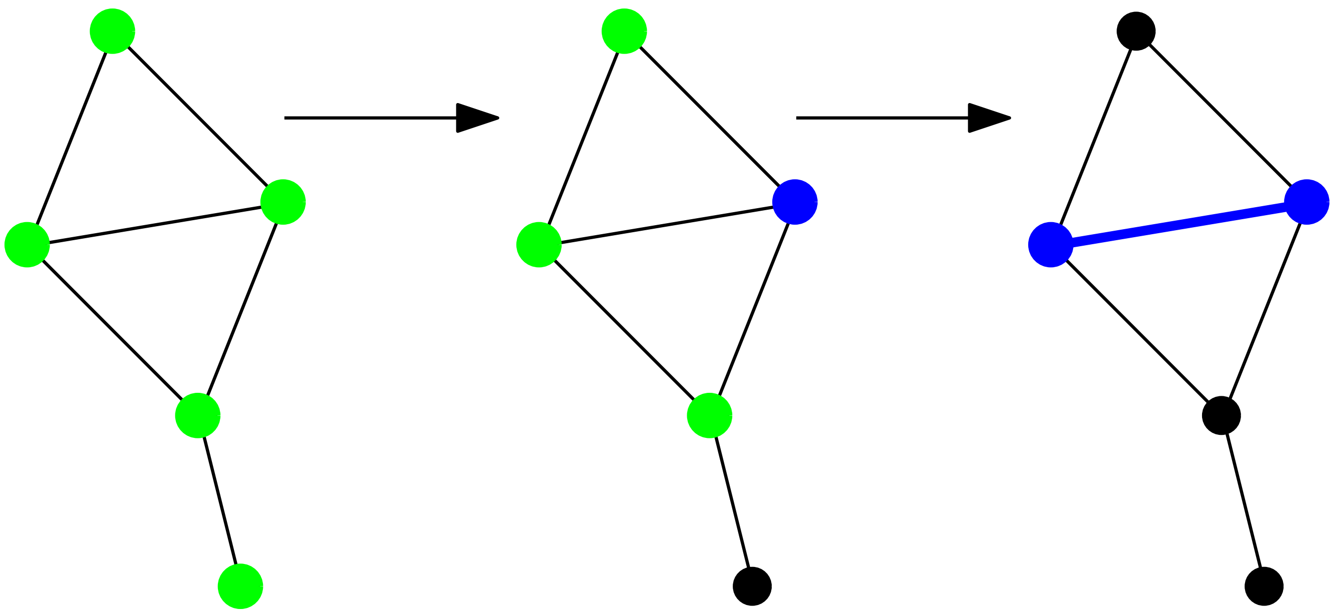

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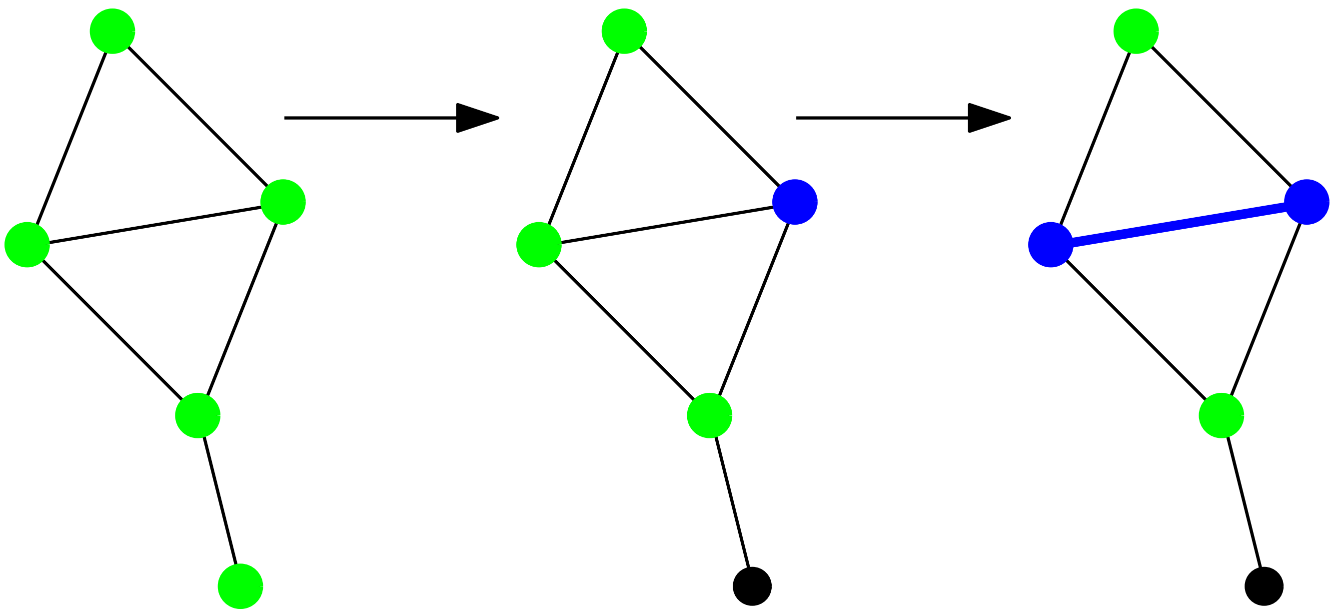

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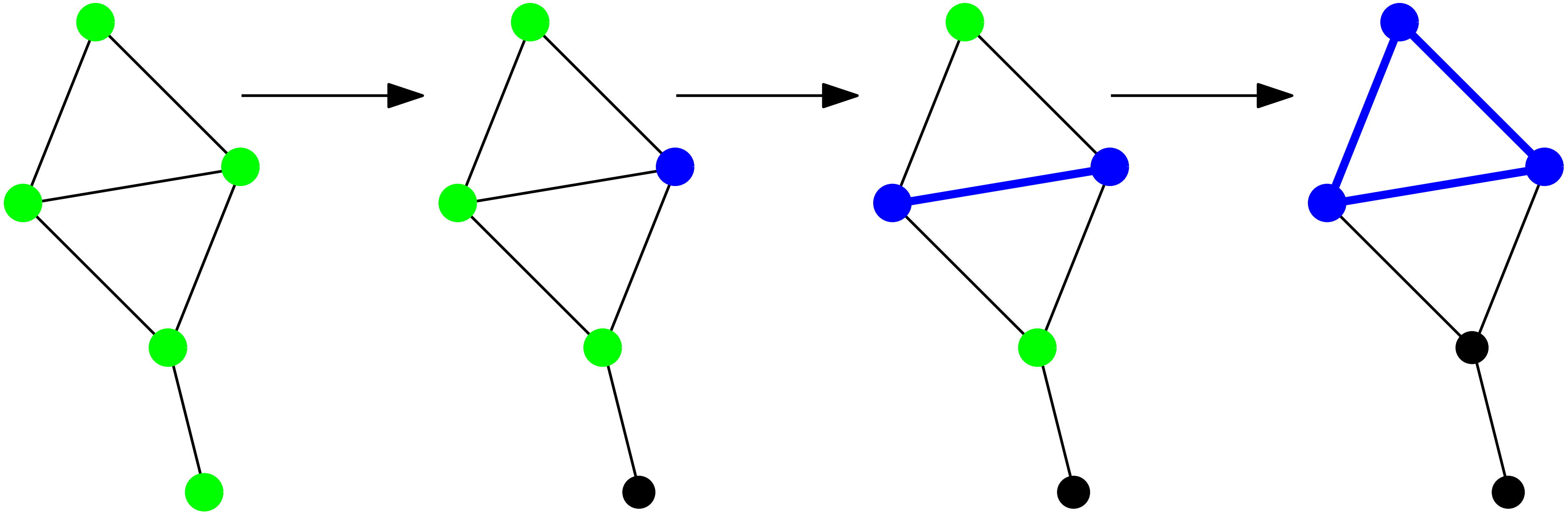

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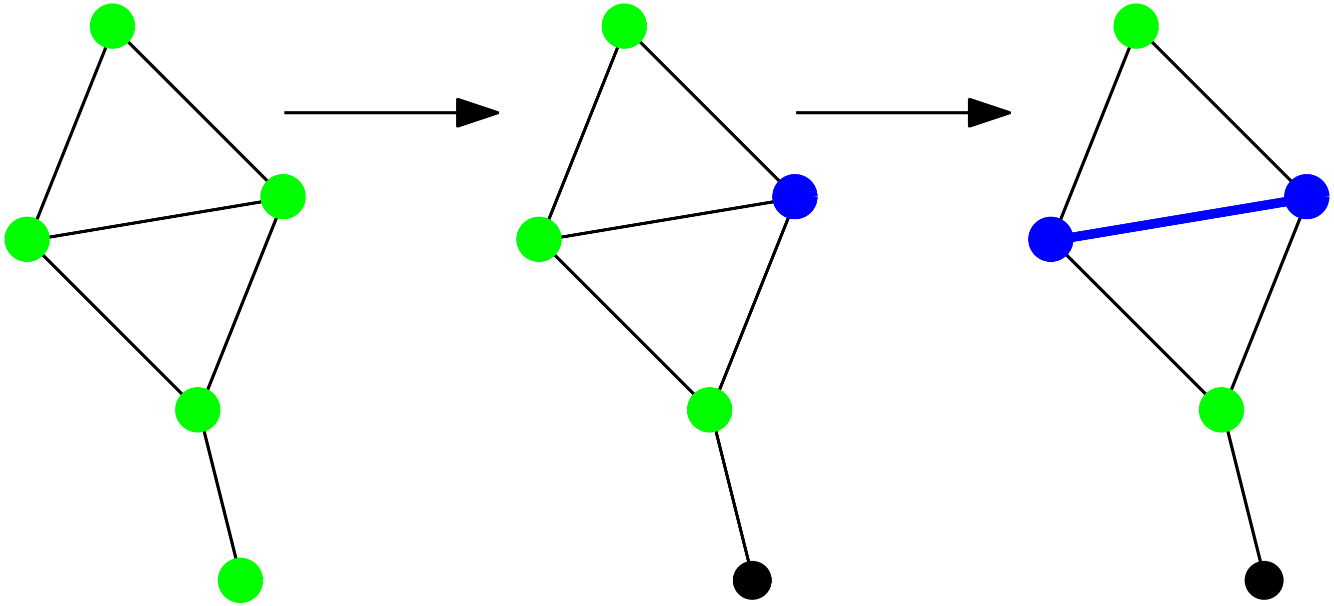

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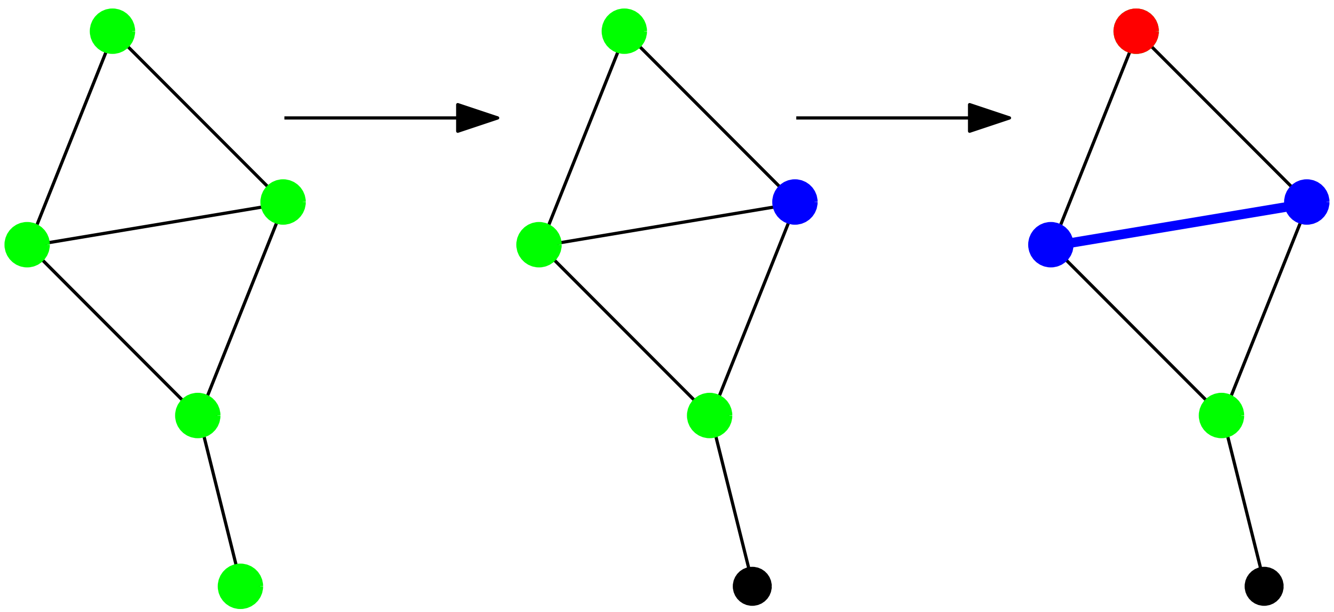

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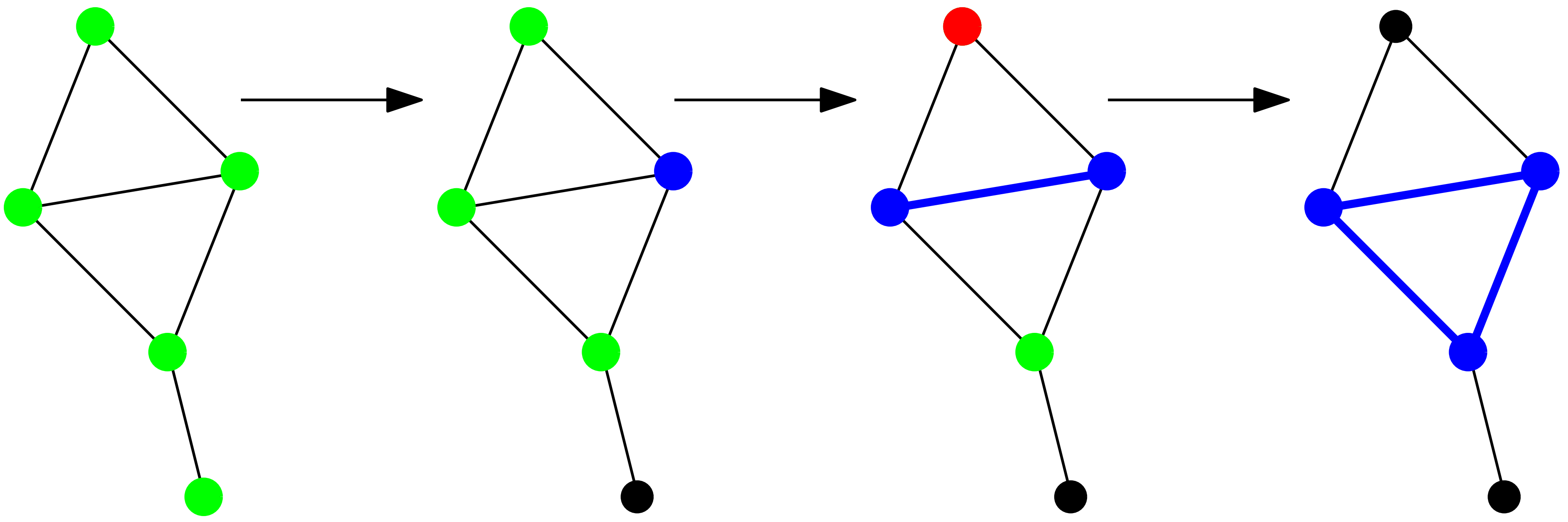

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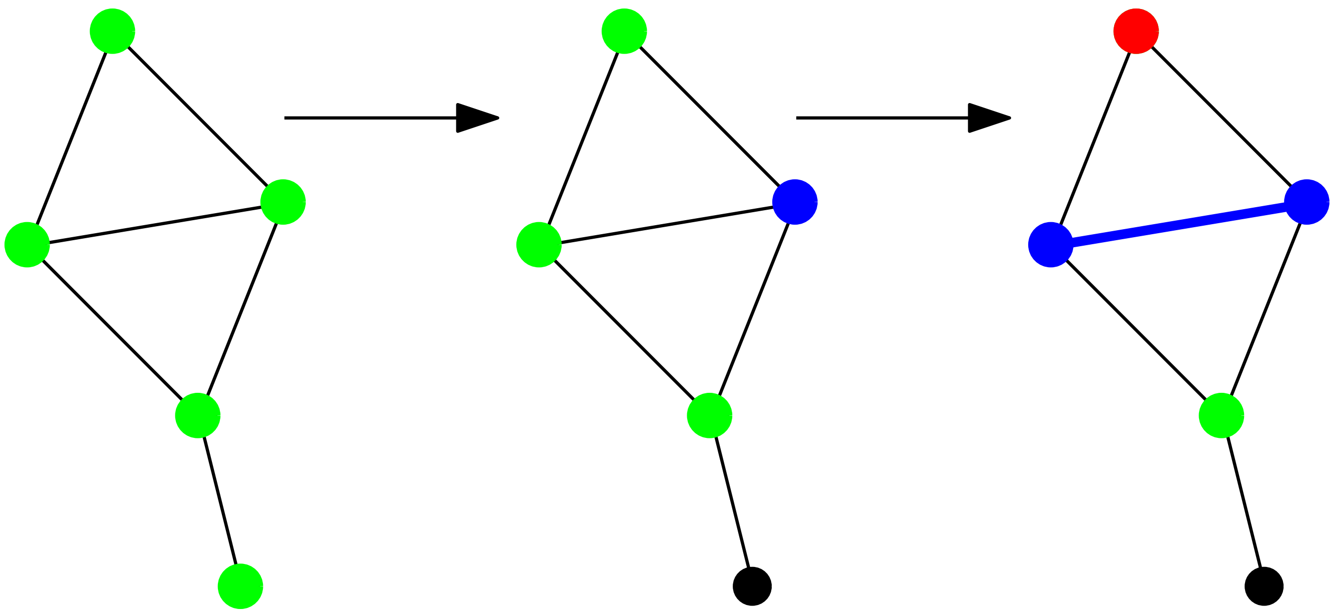

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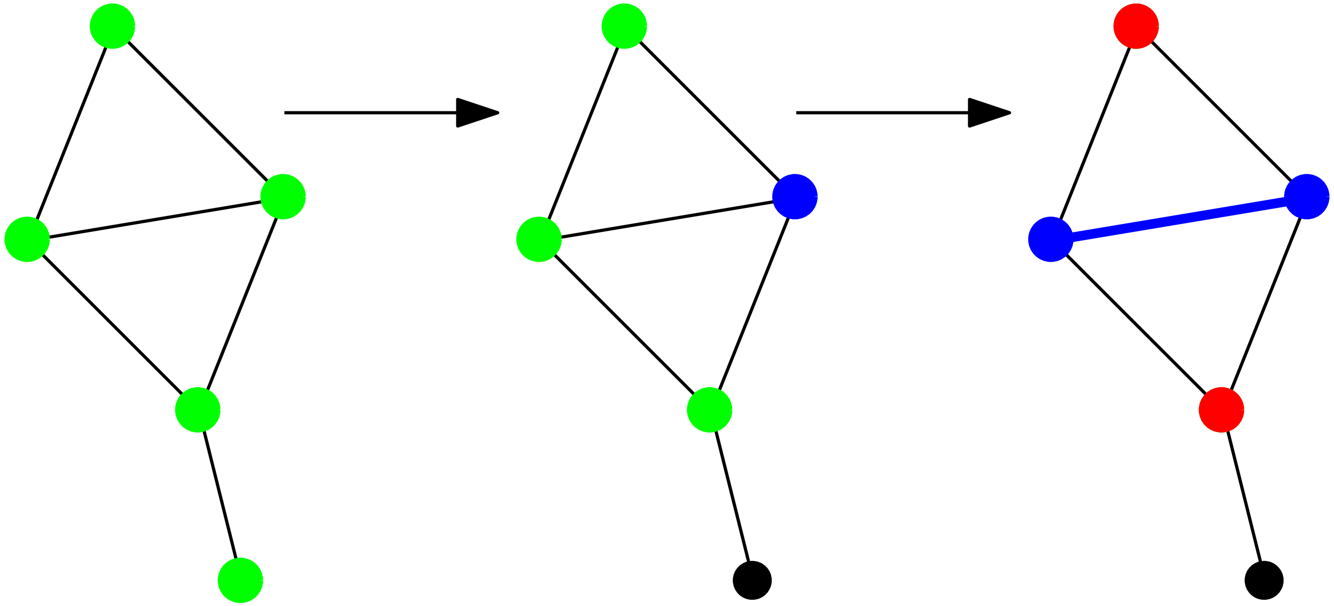

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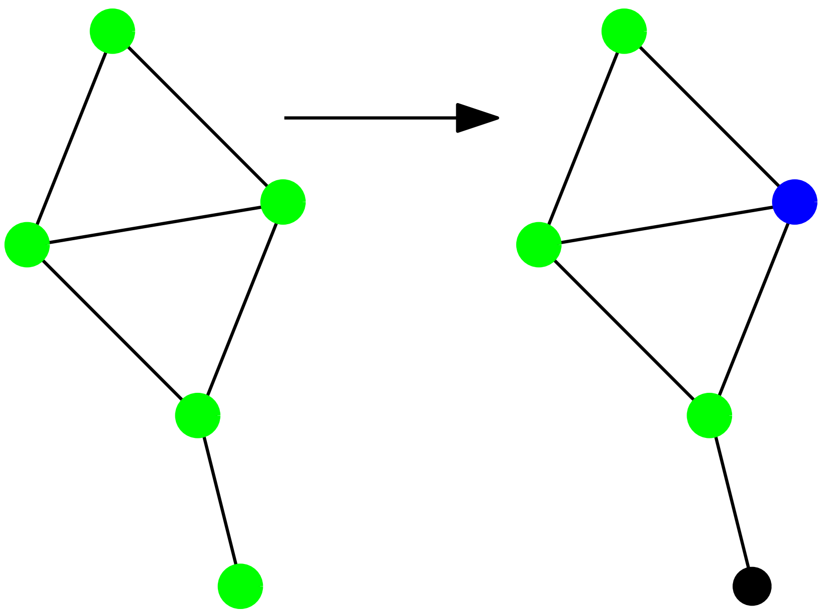

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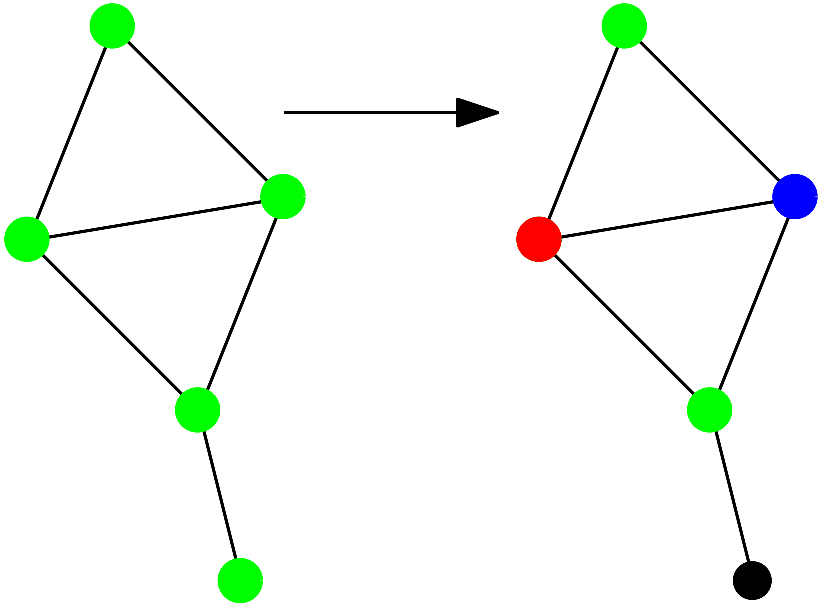

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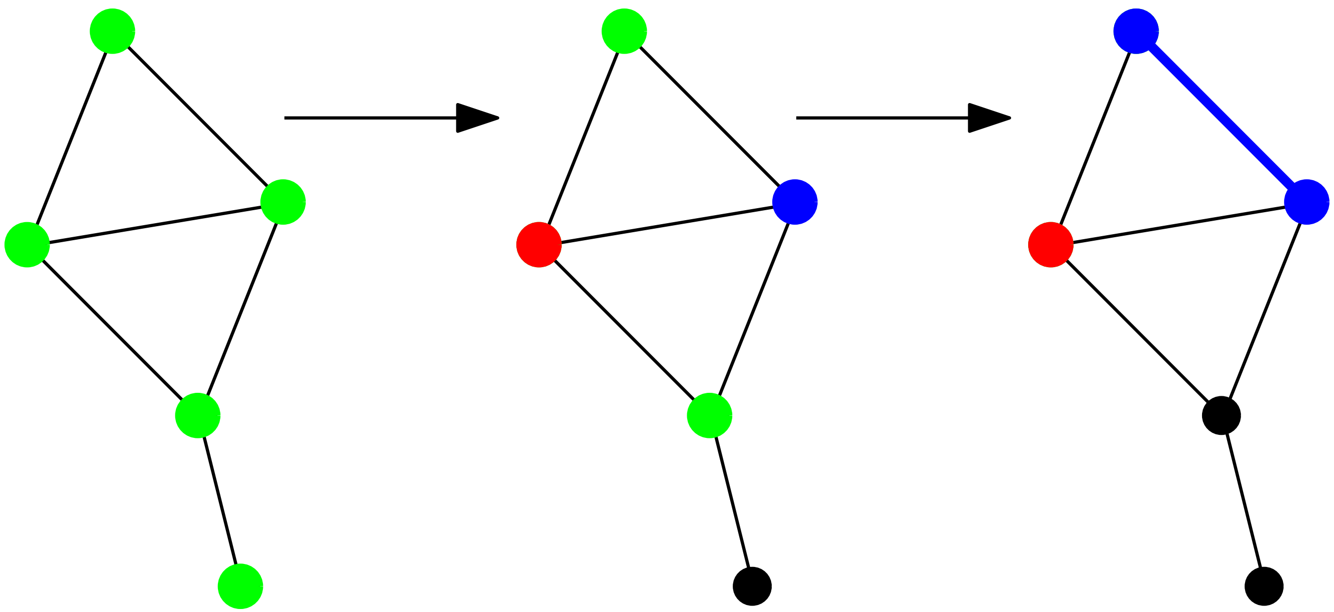

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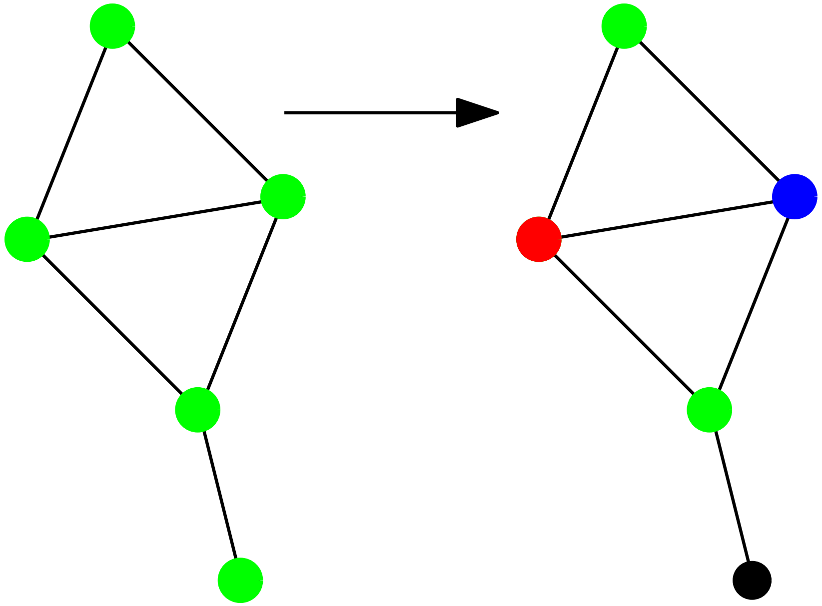

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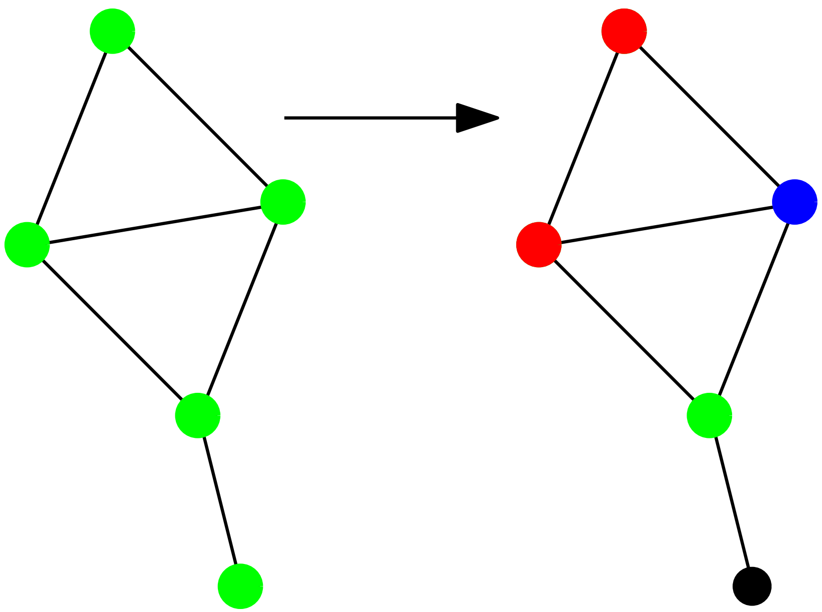

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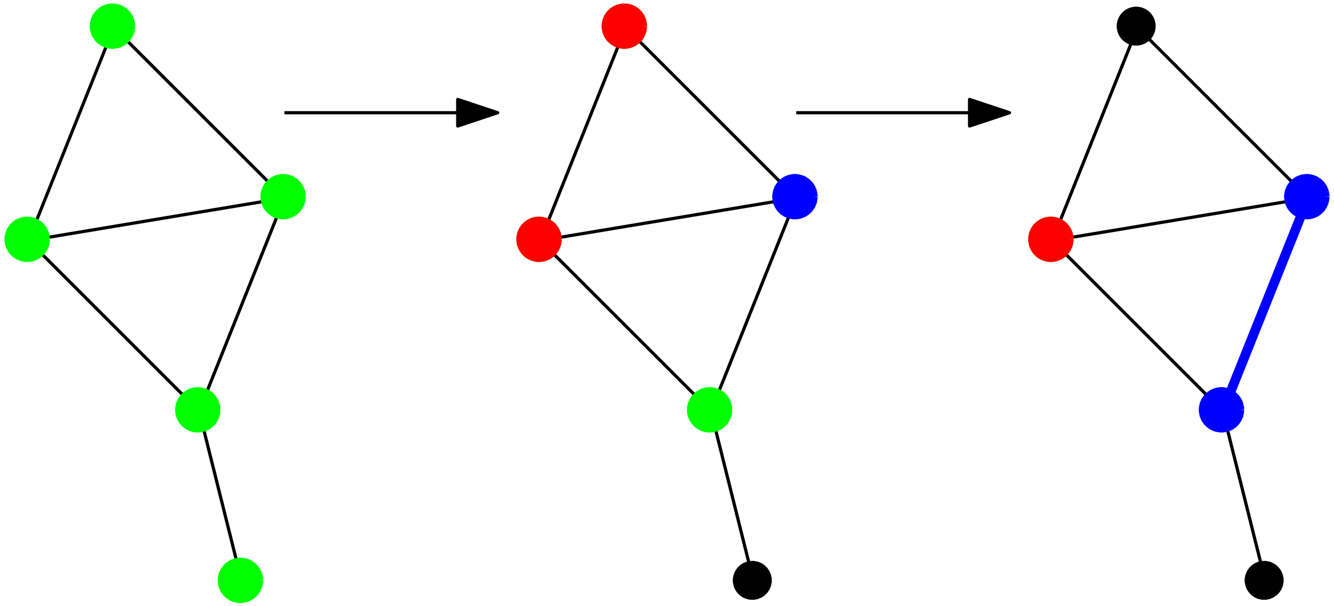

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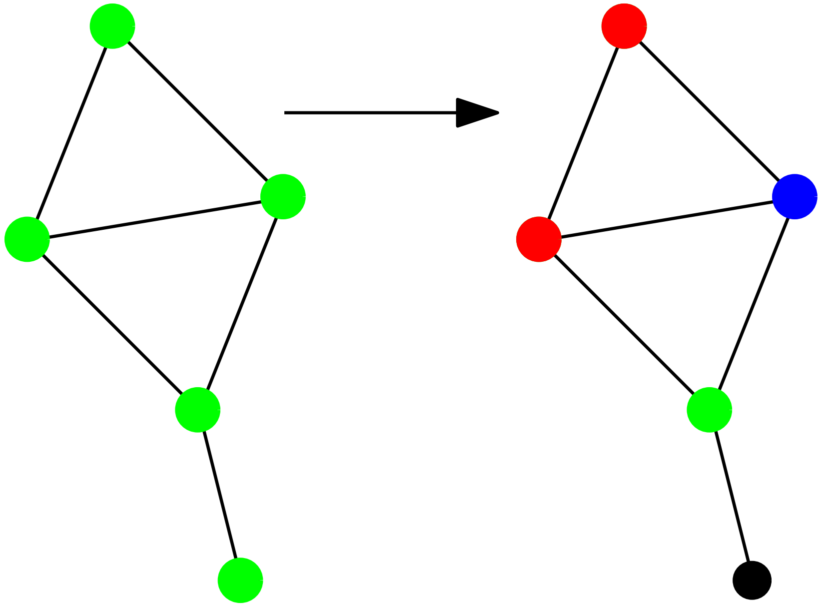

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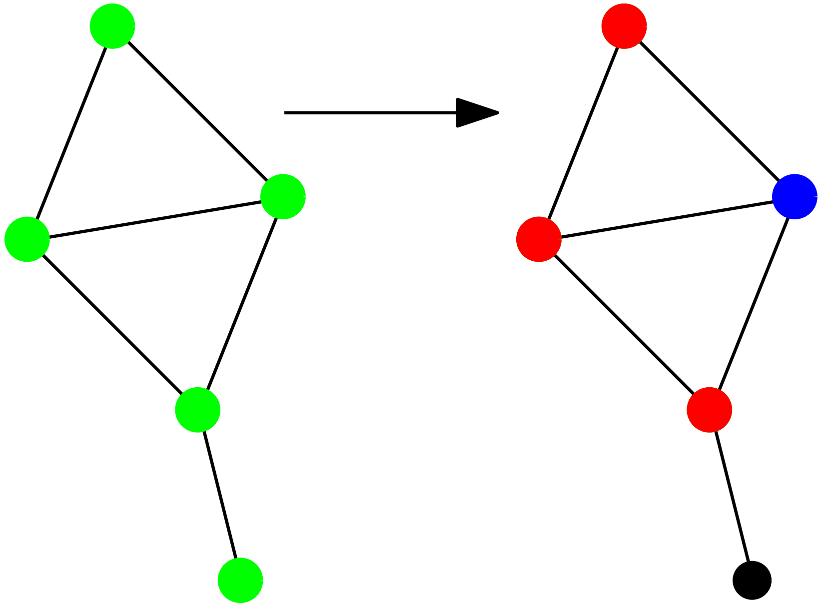

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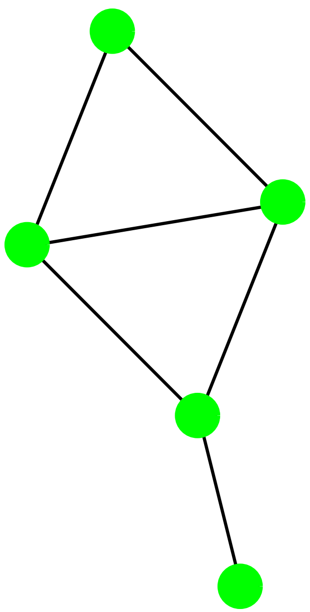

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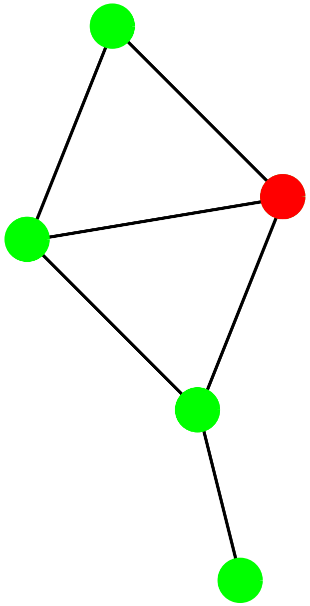


The Bron–Kerbosch Algorithm

proc BronKerbosch(P, R, X)

if $P \cup X = \emptyset$ **then**

    report R as a maximal clique

end if

for each vertex $v \in P$ **do**

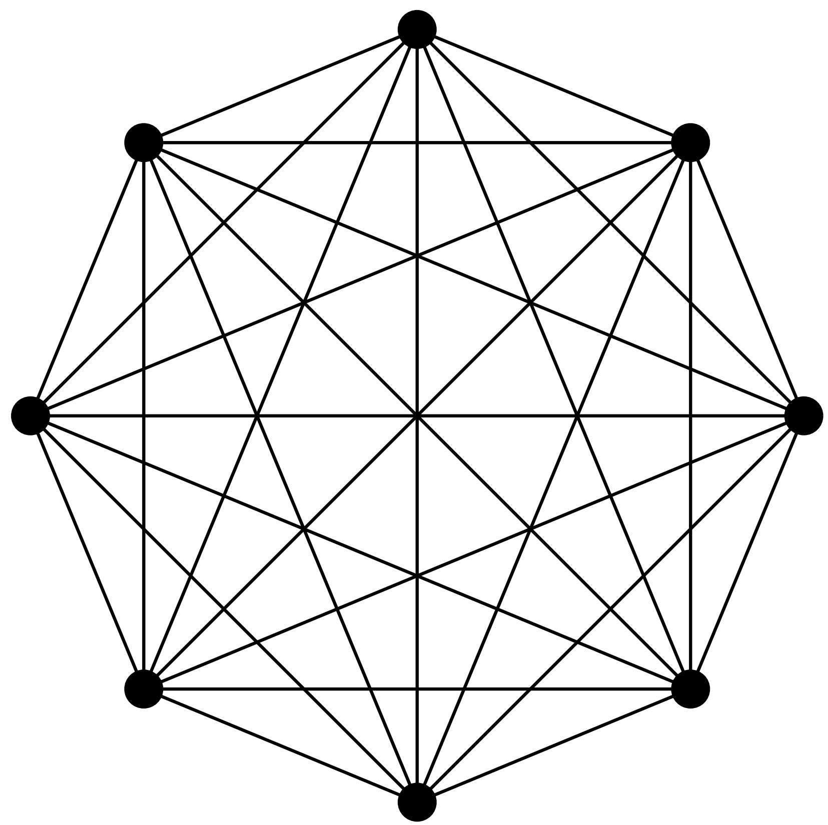
    BronKerbosch($P \cap \Gamma(v)$, $R \cup \{v\}$, $X \cap \Gamma(v)$)

    $P \leftarrow P \setminus \{v\}$

    $X \leftarrow X \cup \{v\}$

end for

The Bron–Kerbosch Algorithm



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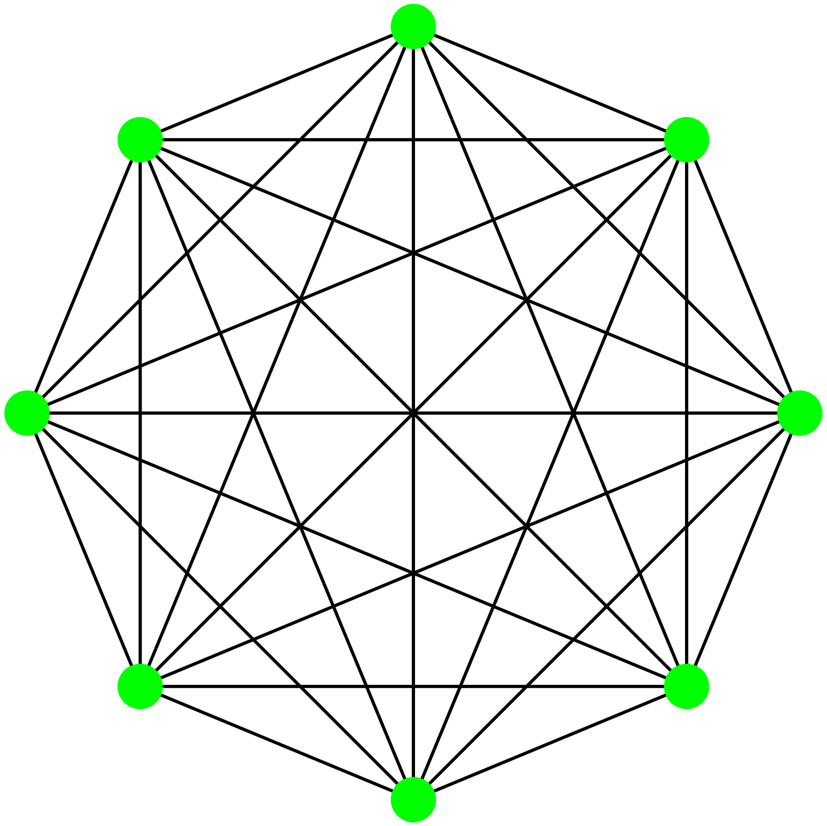

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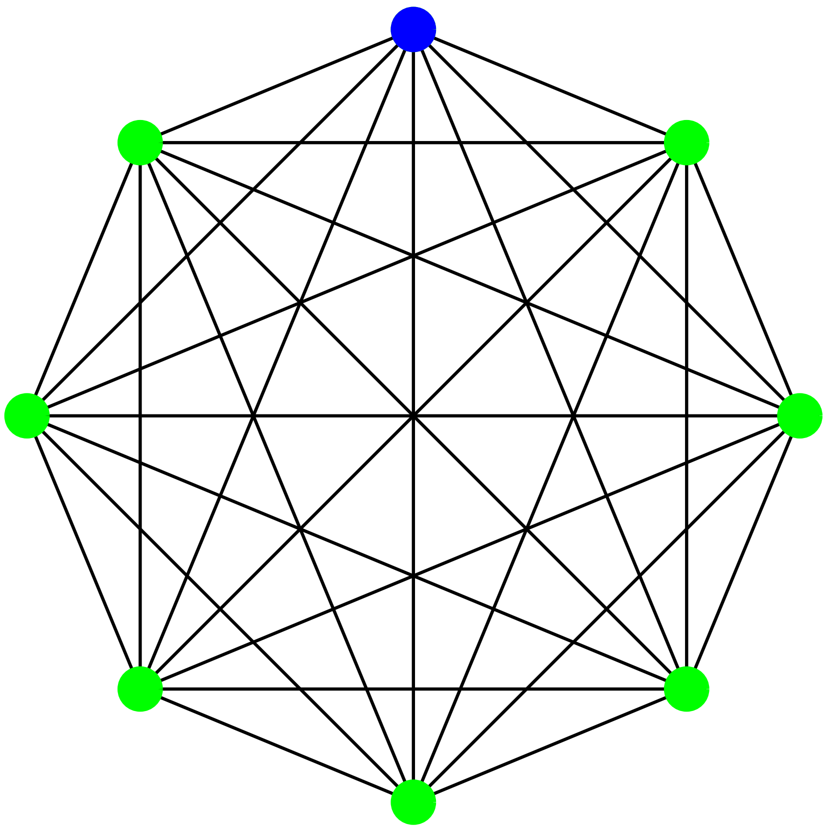

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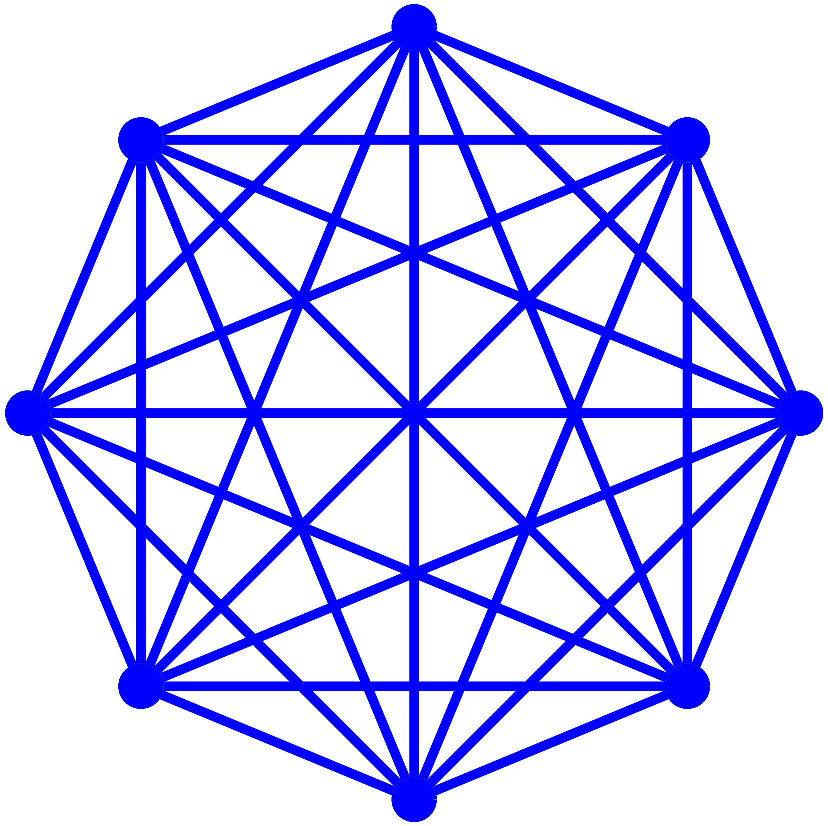

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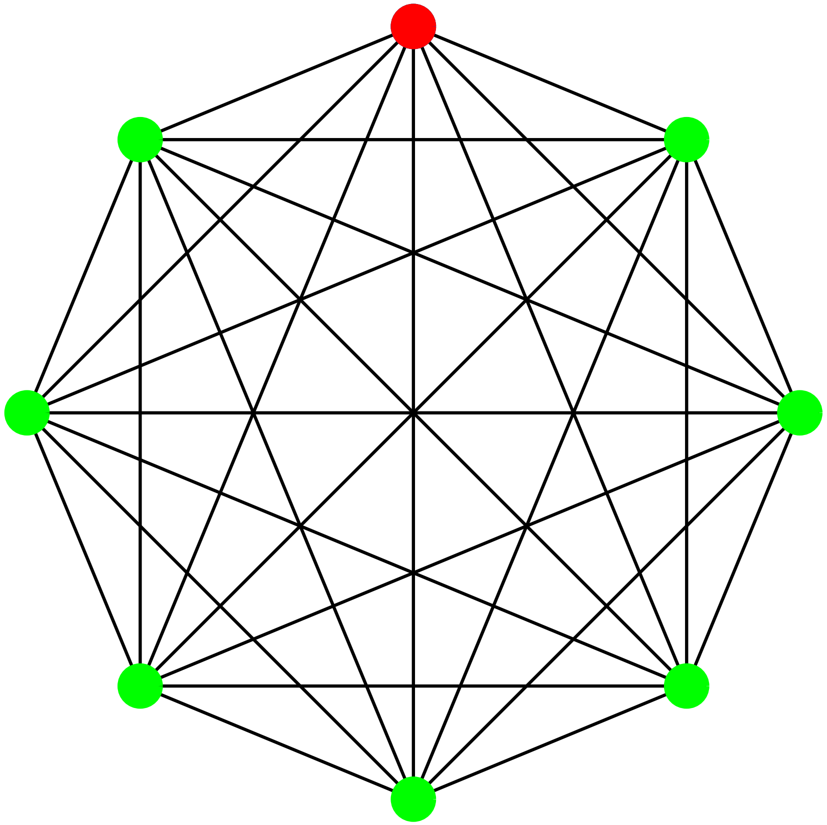

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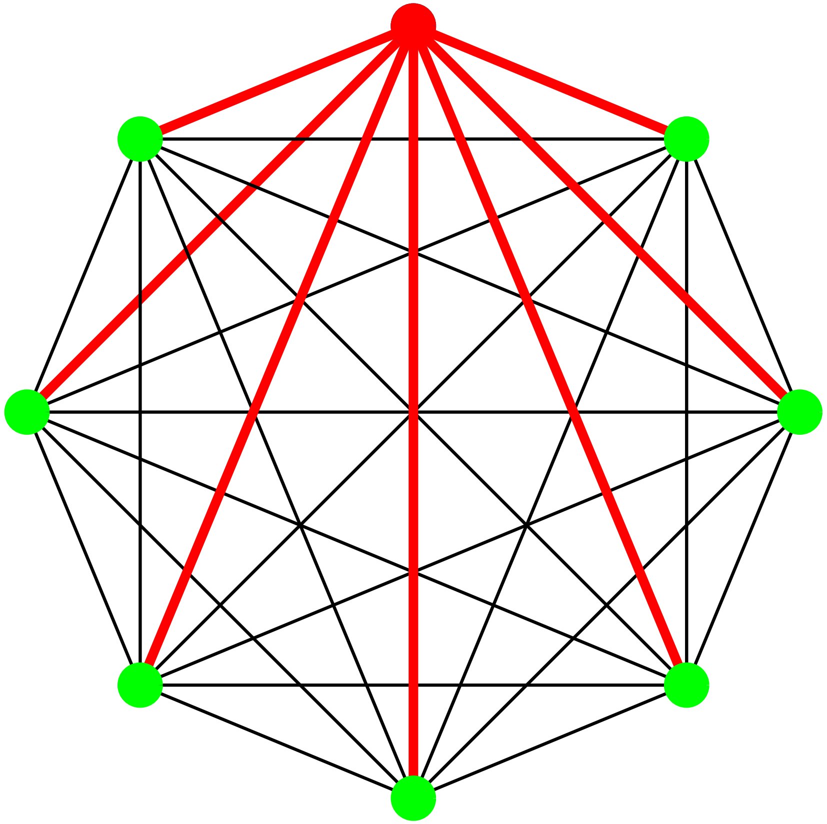

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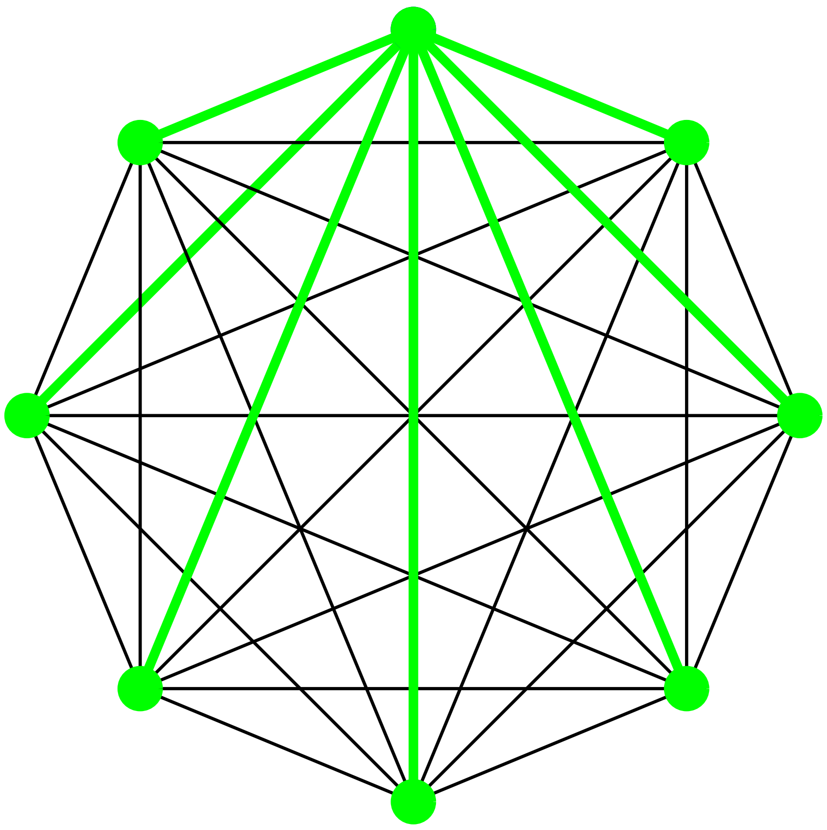

The Bron–Kerbosch Algorithm with Pivoting

proc BronKerboschPivot(P, R, X)

if $P \cup X = \emptyset$ **then**
    report R as a maximal clique
end if
choose a pivot $u \in P \cup X$
for each vertex $v \in P \setminus \Gamma(u)$ **do**
    BronKerboschPivot($P \cap \Gamma(v)$, $R \cup \{v\}$, $X \cap \Gamma(v)$)
    $P \leftarrow P \setminus \{v\}$
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end for

The Bron–Kerbosch Algorithm with Pivoting

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The Bron–Kerbosch Algorithm with Pivoting

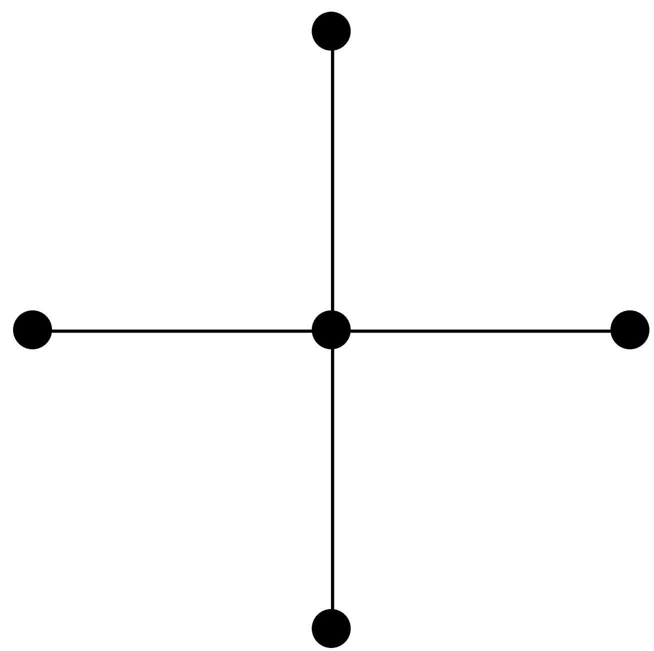
$$T(n) \leq \max_k \{kT(n-k)\} + O(n^2)$$

The Bron–Kerbosch Algorithm with Pivoting

$$T(n) \leq \max_k \{kT(n-k)\} + O(n^2)$$

$$T(n) = O(3^{n/3})$$

The Bron–Kerbosch Algorithm



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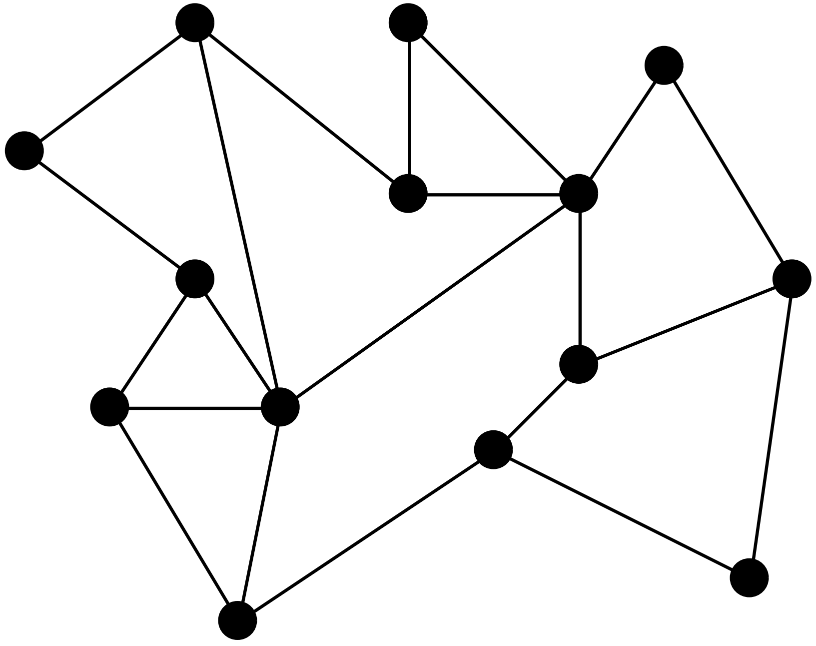

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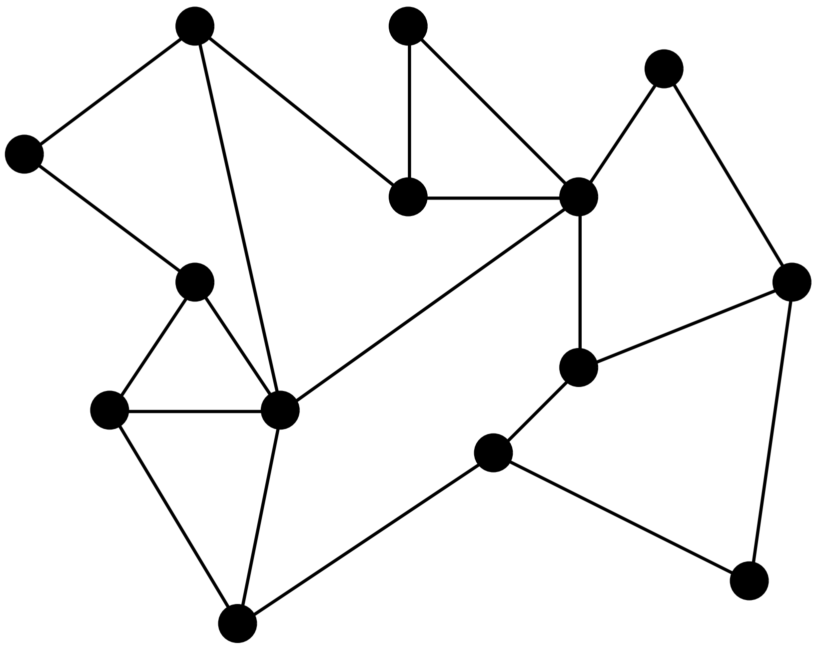


All cliques in planar graphs may be listed in time $O(n)$

Chiba and Nishizeki (1985), Chrobak and Eppstein (1991)

The Bron–Kerbosch Algorithm

Want to characterize the running time with a parameter.

Let p be our parameter of choice.

An algorithm is *fixed-parameter tractable* with parameter p if it has running time

$$f(p)n^{O(1)}$$

The key is to avoid things like n^p.

Parameterize on Sparsity

Parameterize on Sparsity

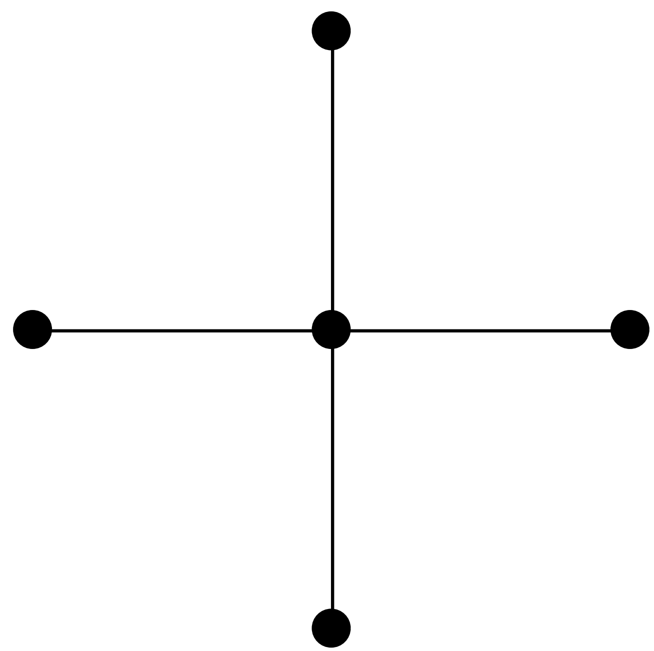
degeneracy:

Parameterize on Sparsity

degeneracy:

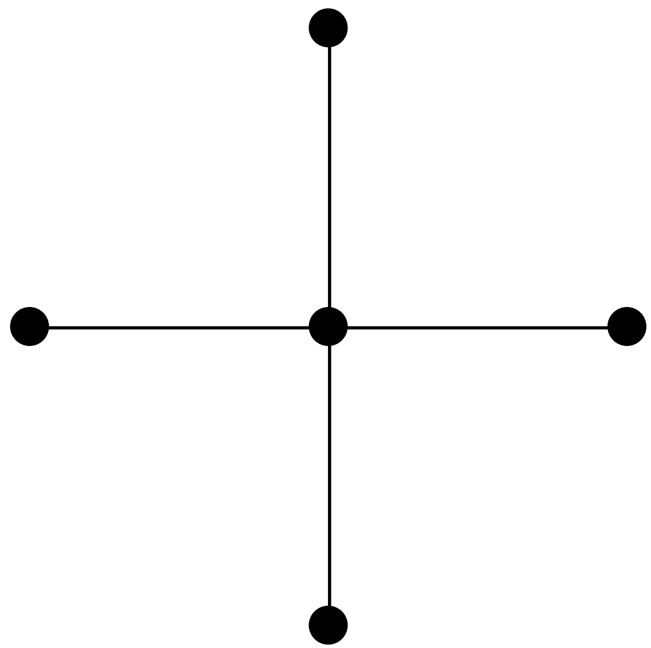
The minimum integer d such that every subgraph of G has a vertex of degree d or less.

Degeneracy



Degeneracy

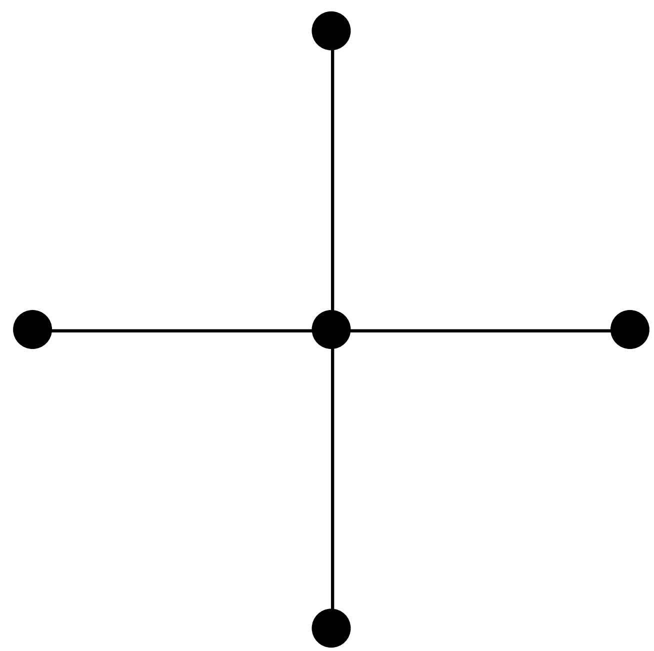
$d = 1$

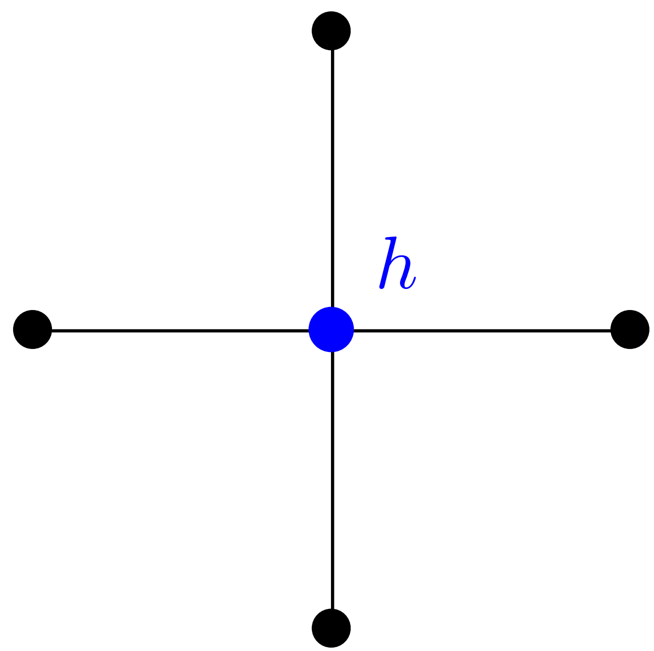


Degeneracy

degeneracy:

The minimum integer d such that there is an ordering of the vertices where each vertex has at most d neighbors later in the ordering.

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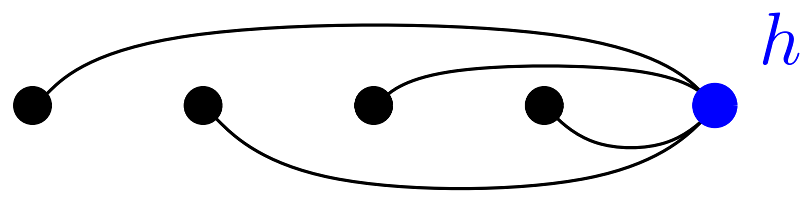
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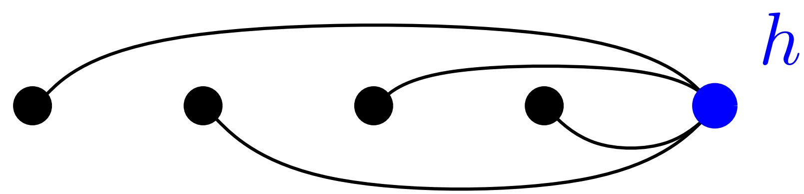
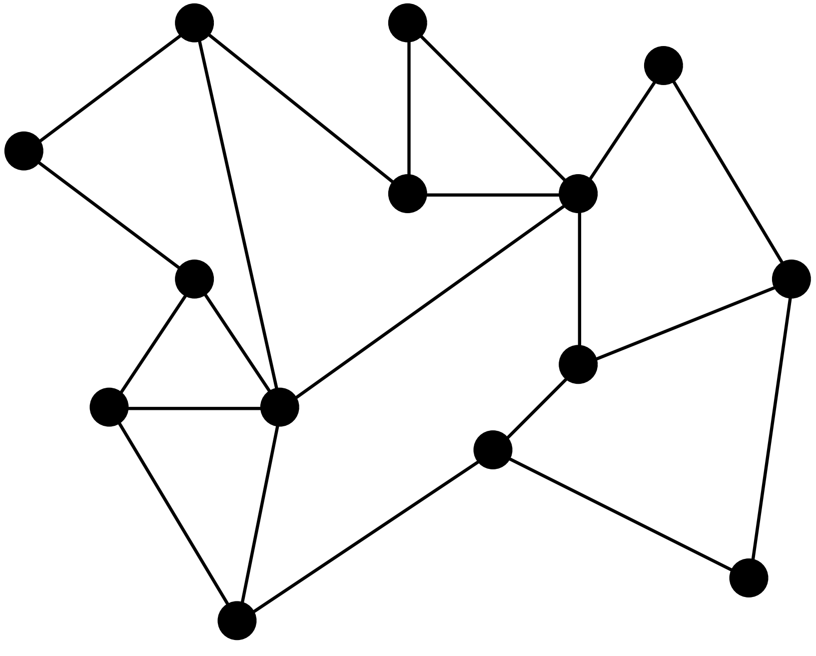
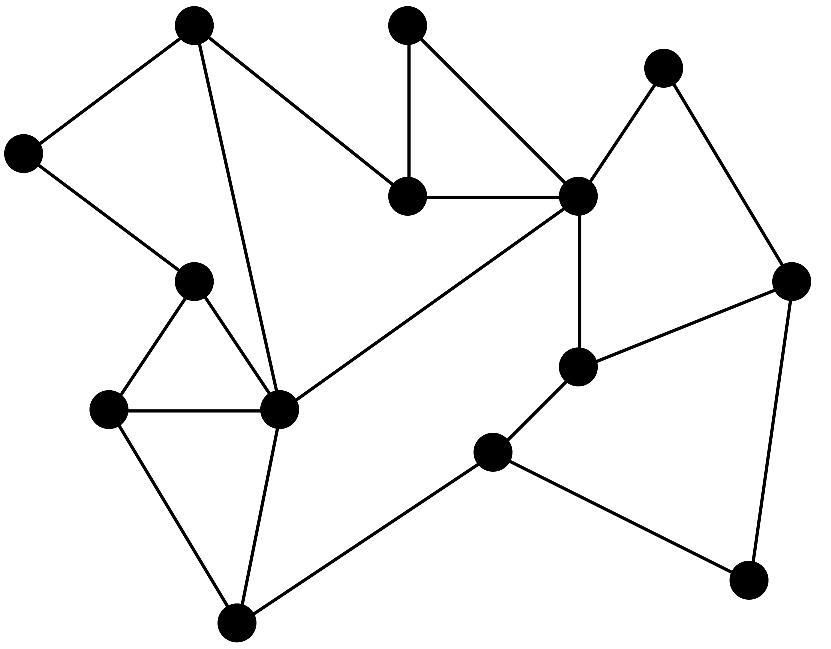
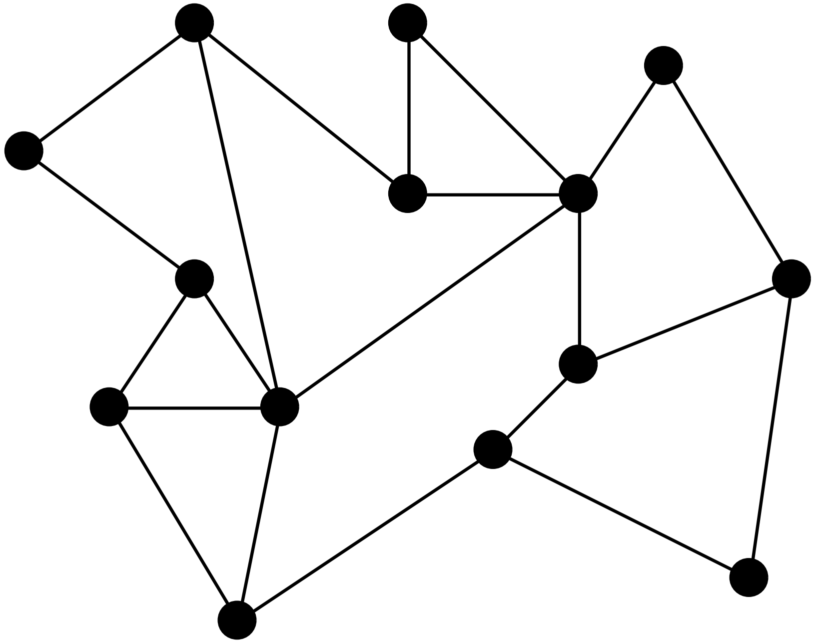


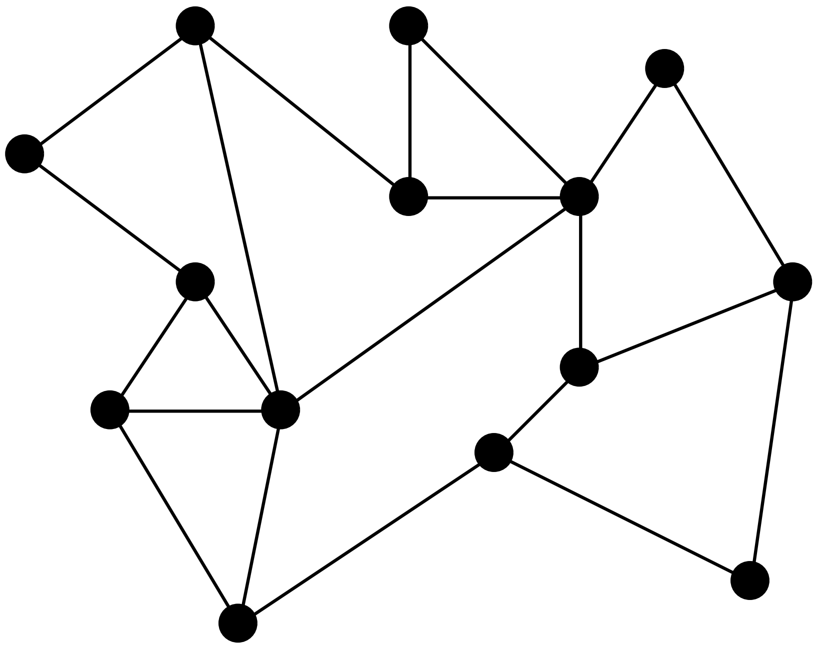


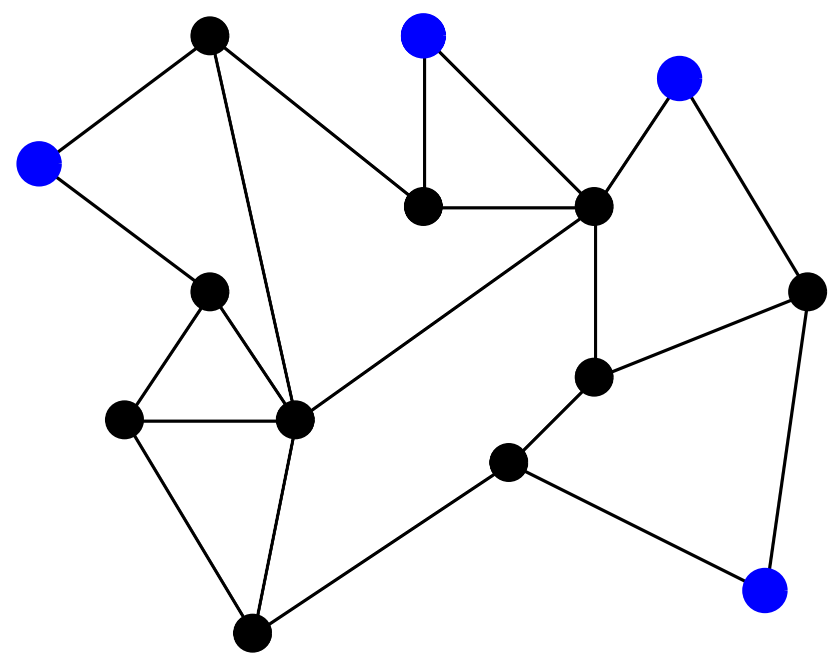
Planar graphs have degeneracy at most 5

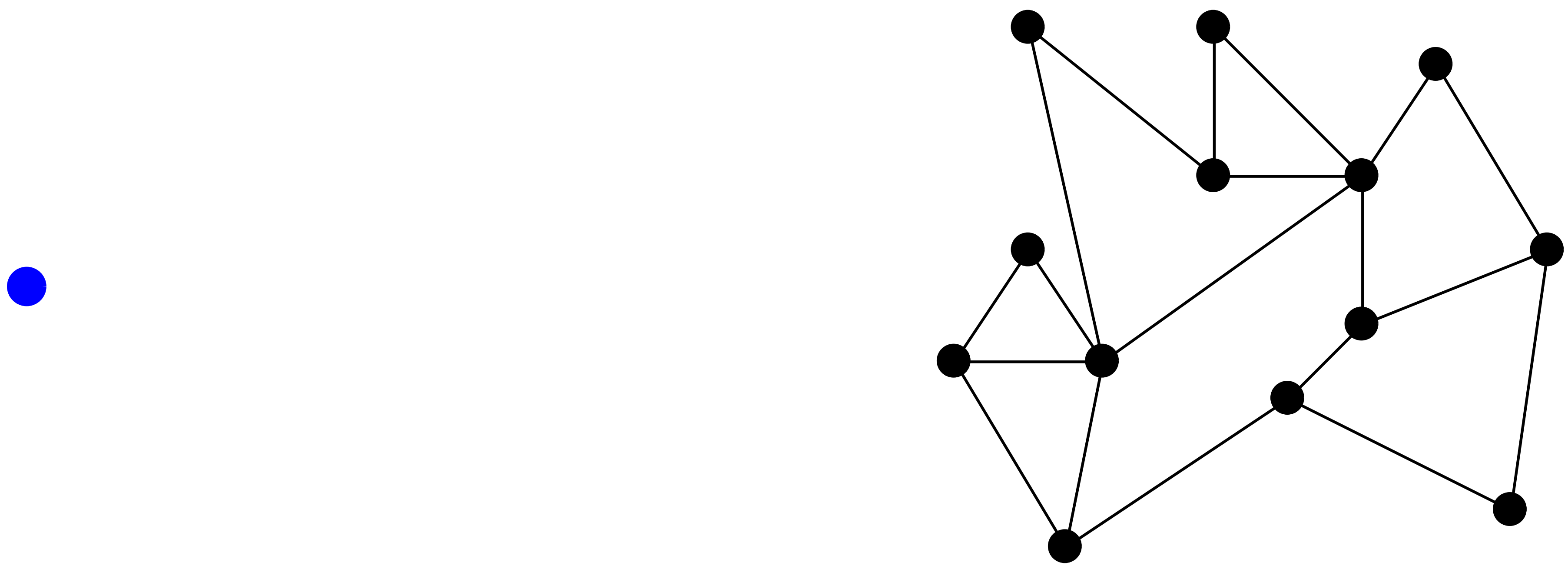


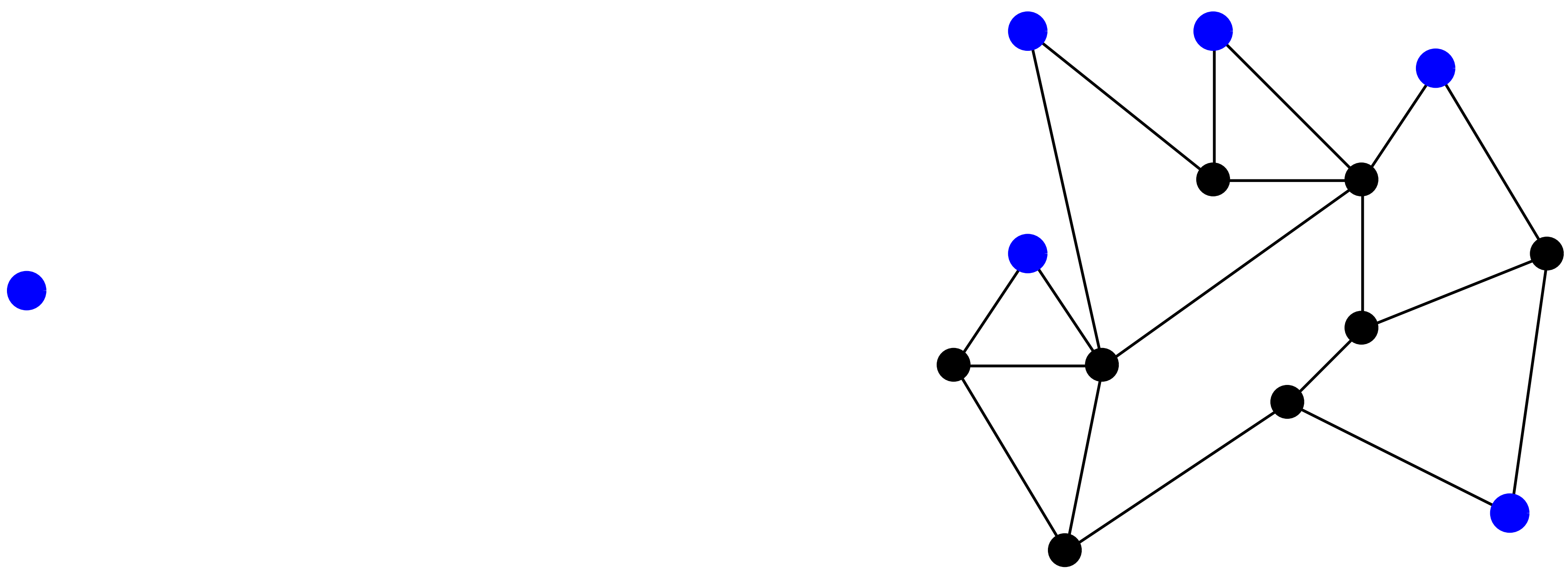
Degeneracy is easy to compute

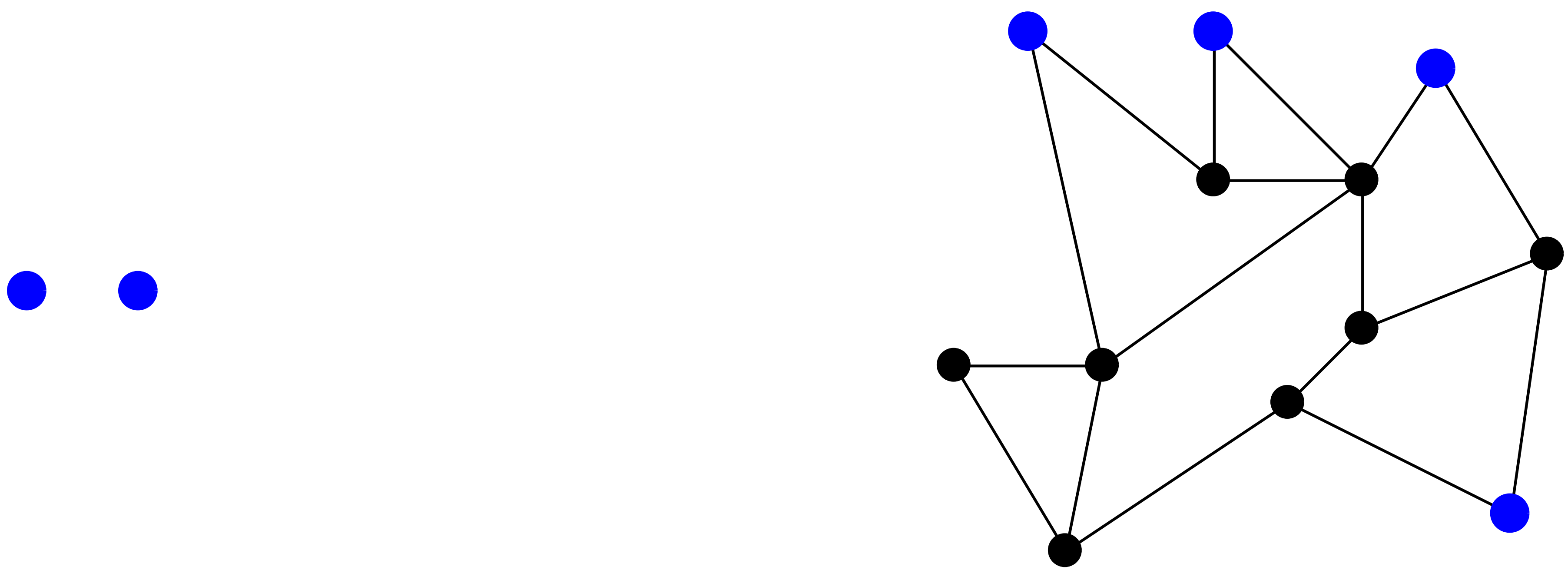


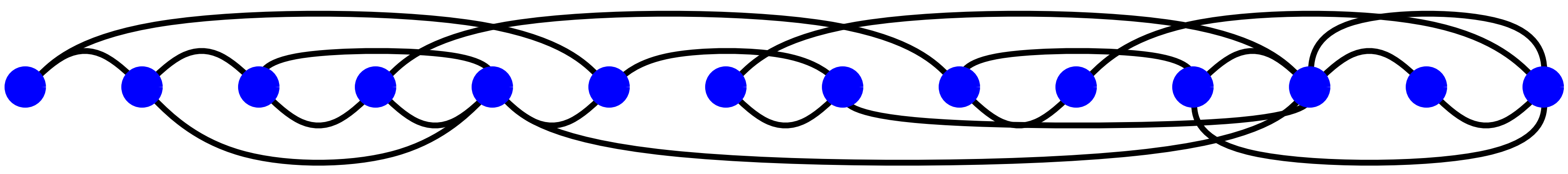












d-degenerate graphs...

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cannot contain cliques with more than $d+1$ vertices

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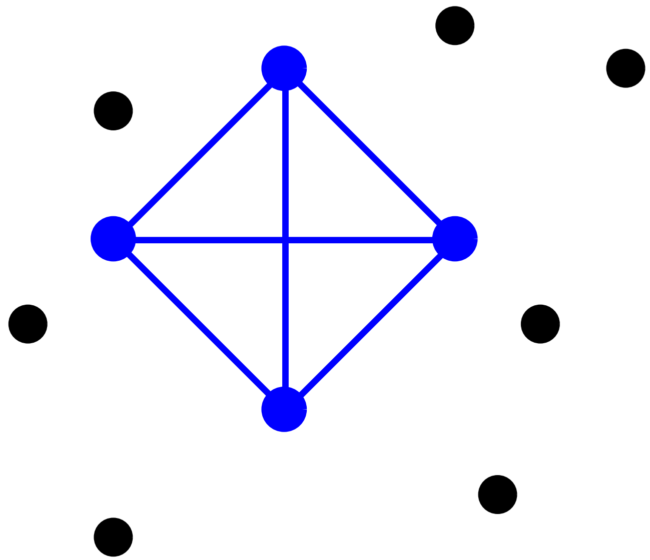

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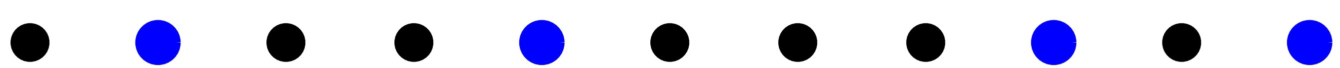

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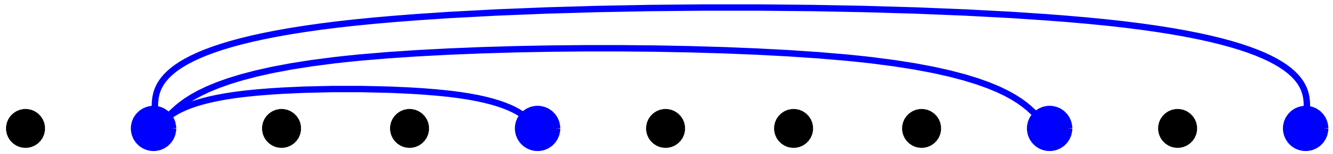

d-degenerate graphs...

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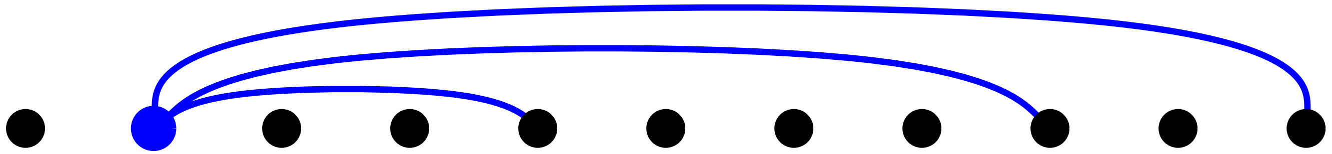

A few more facts about degeneracy...

Degeneracy is within a constant factor of other popular sparsity measures.

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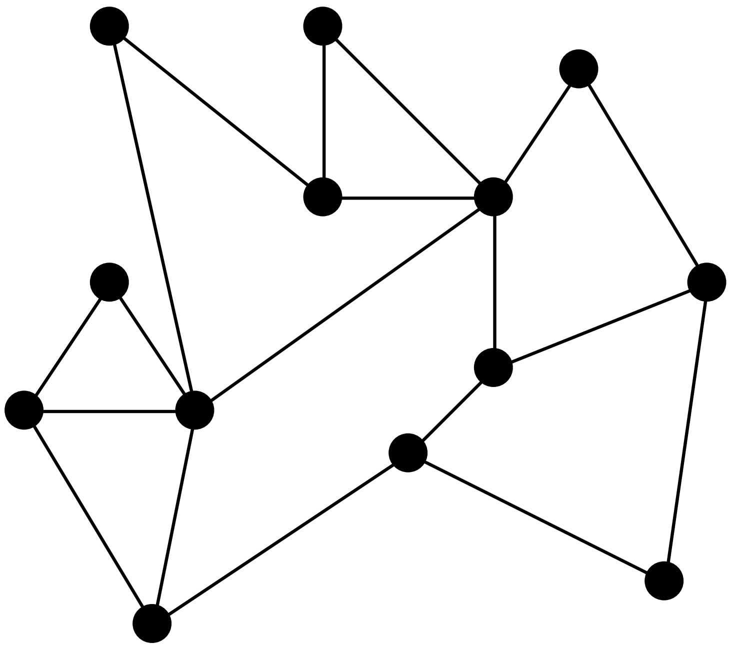
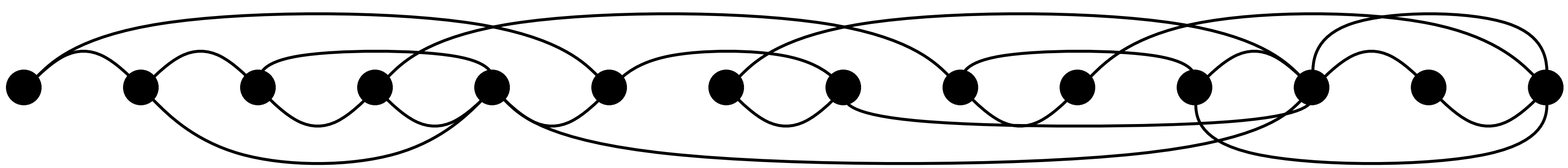
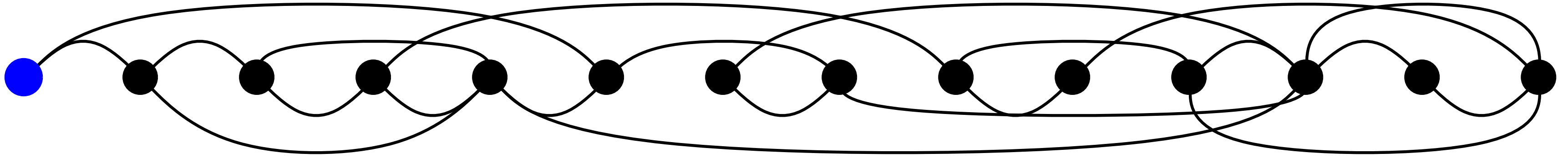
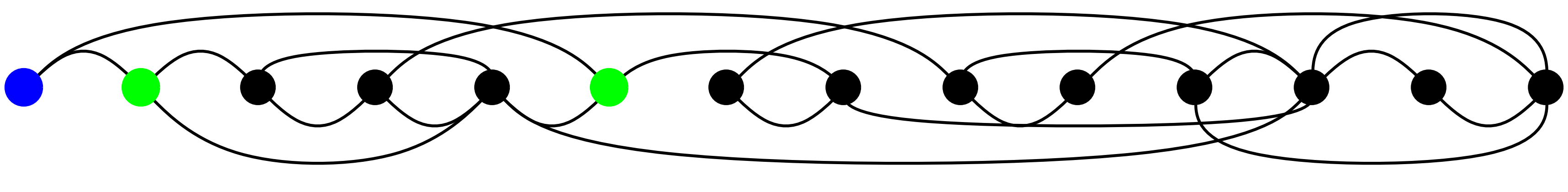
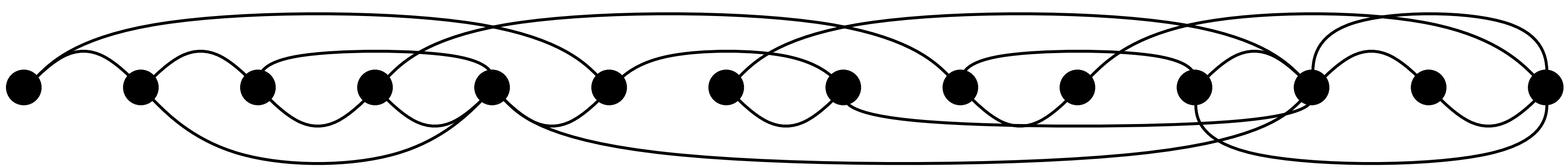
Graphs generated by the preferential attachment mechanism of Barabási and Albert have low degeneracy.

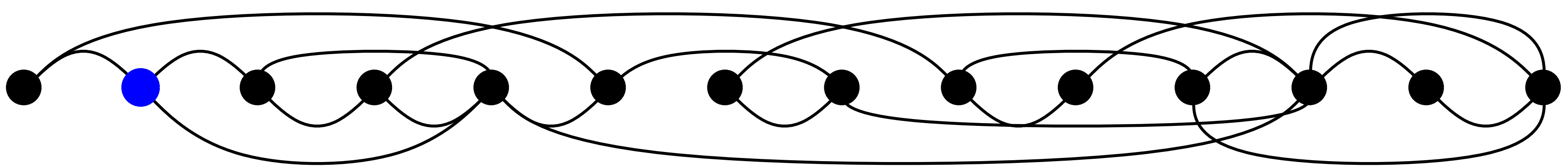


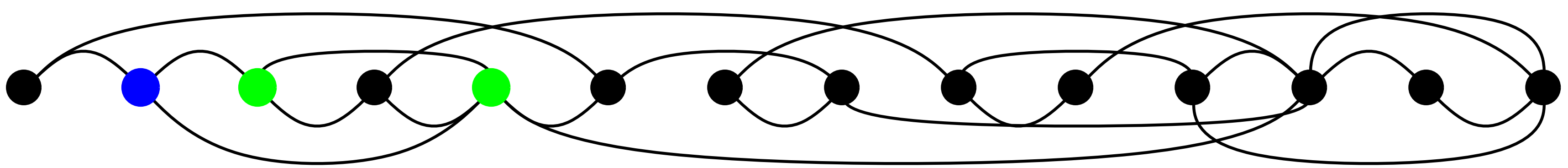


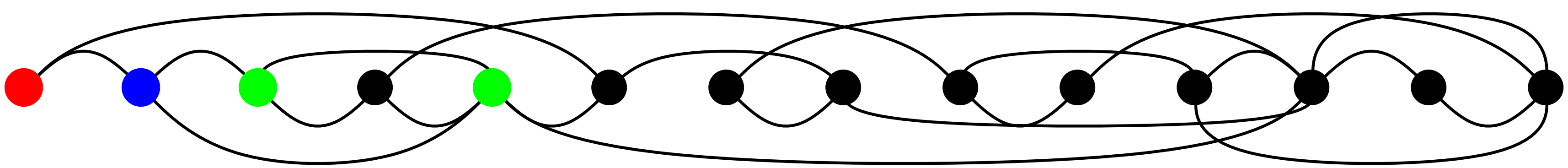












proc BronKerboschDegeneracy(V, E)

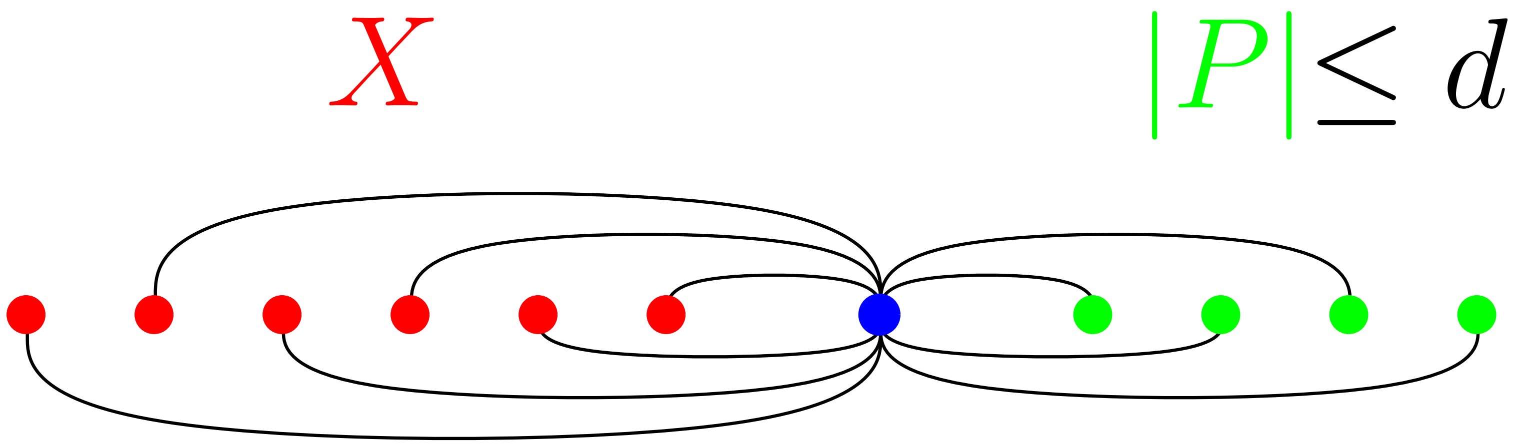
for each vertex v_i in a degeneracy ordering v_0, v_1, v_2, $\ldots$ of (V, E) **do**

  $P \leftarrow \Gamma(v_i) \cap \{v_{i+1}, \ldots, v_{n-1}\}$

  $X \leftarrow \Gamma(v_i) \cap \{v_0, \ldots, v_{i-1}\}$

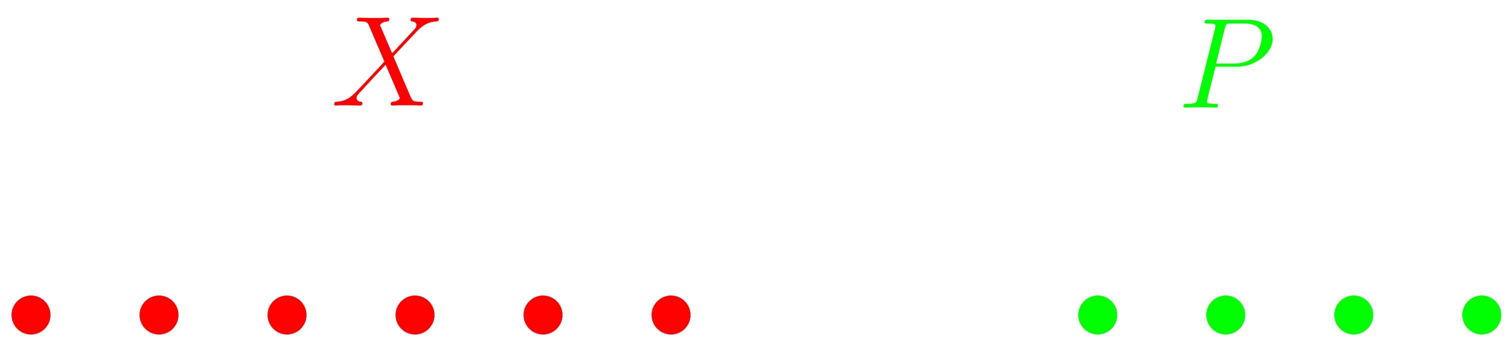
  BronKerboschPivot(P, $\{v_i\}$, X)

end for


X
$|P| \leq d$

Computing the pivot

Pick $u \in X \cup P$ that maximizes $|P \cap \Gamma(u)|$.



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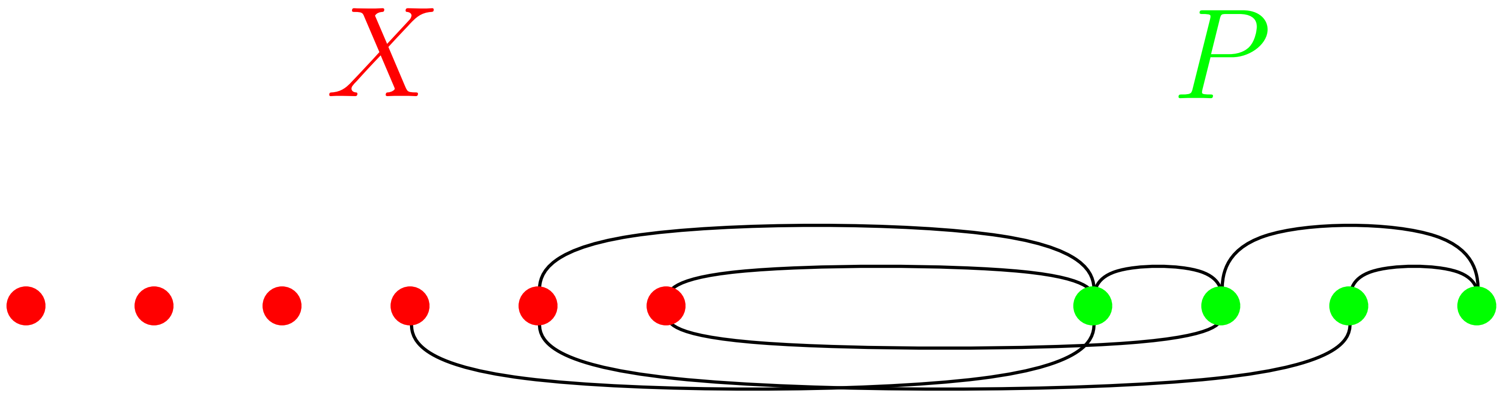

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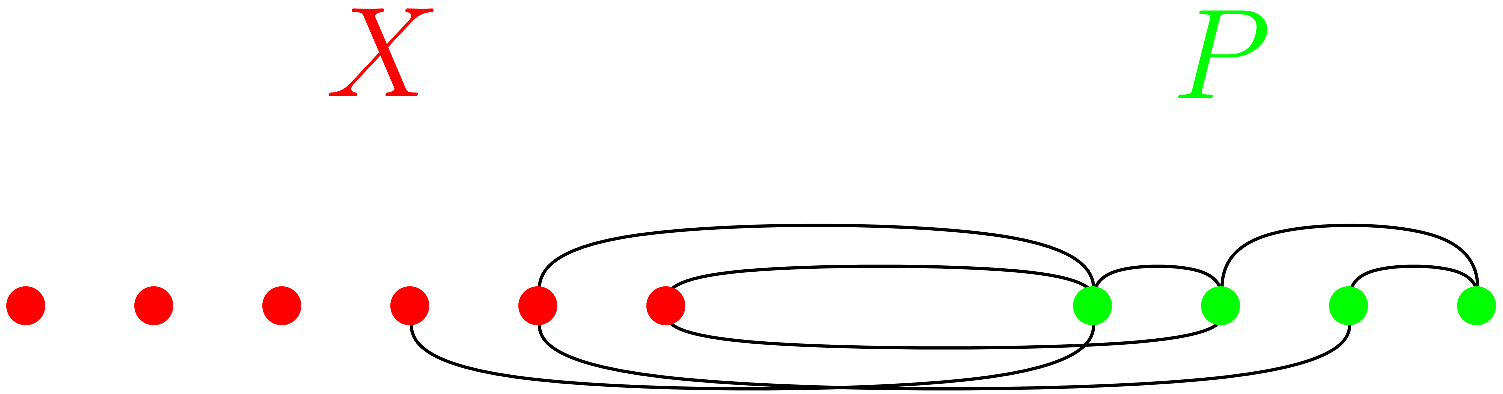

$$O(|P|(|X| + |P|))$$

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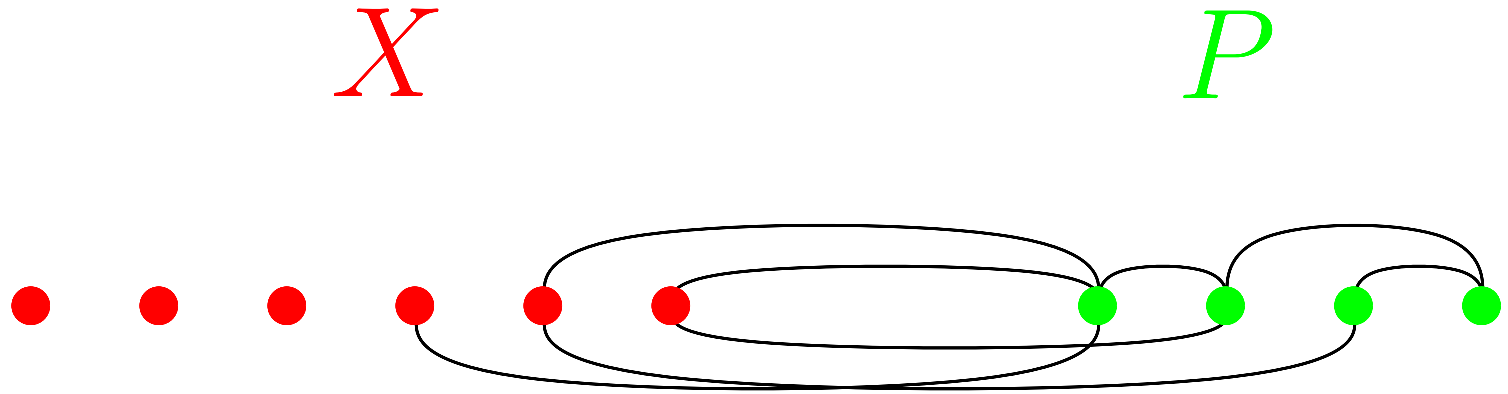

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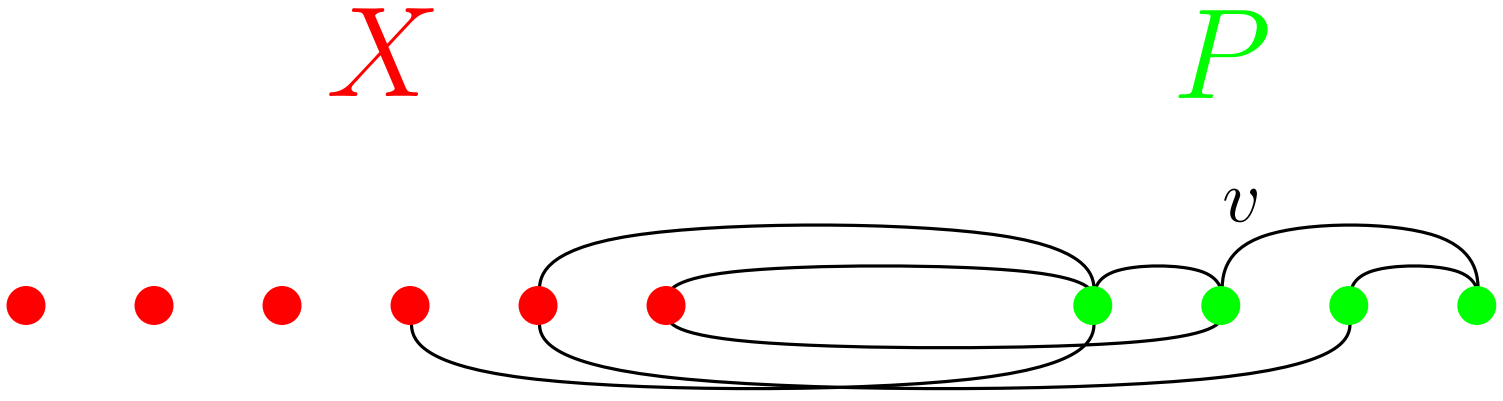

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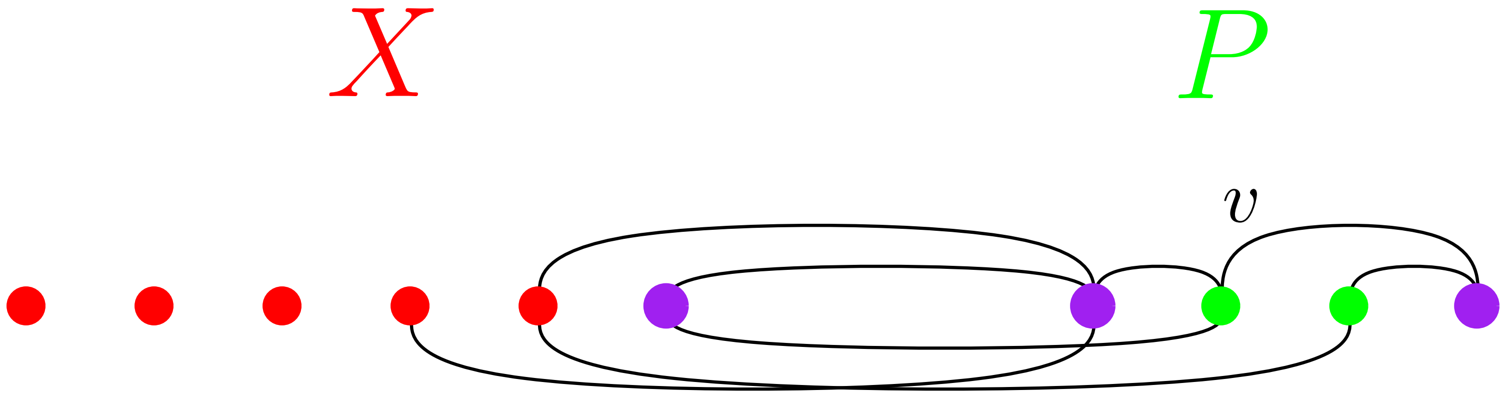

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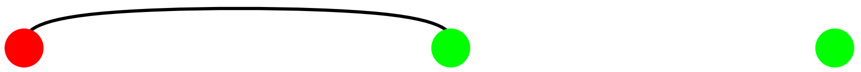

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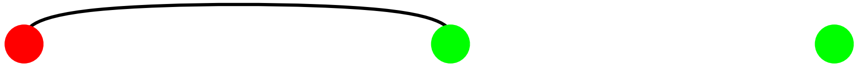

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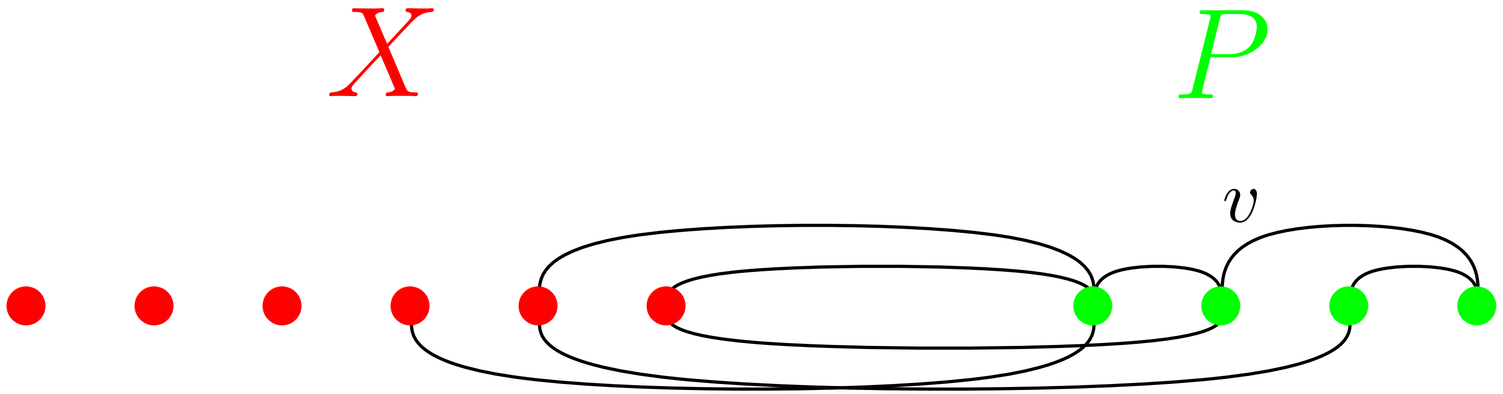

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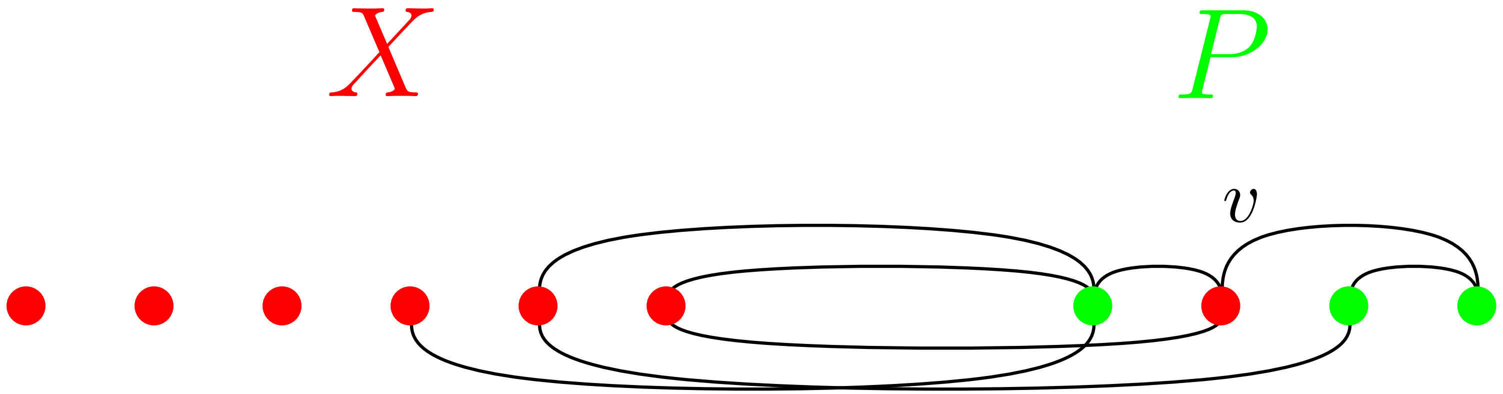

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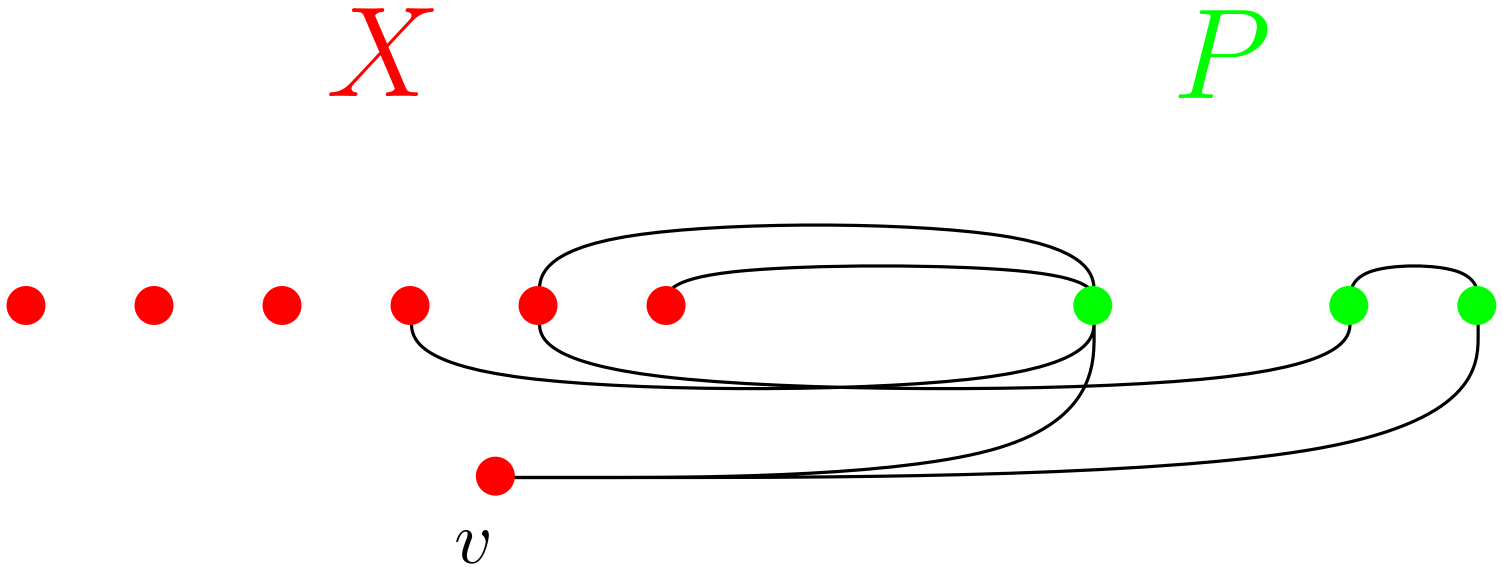

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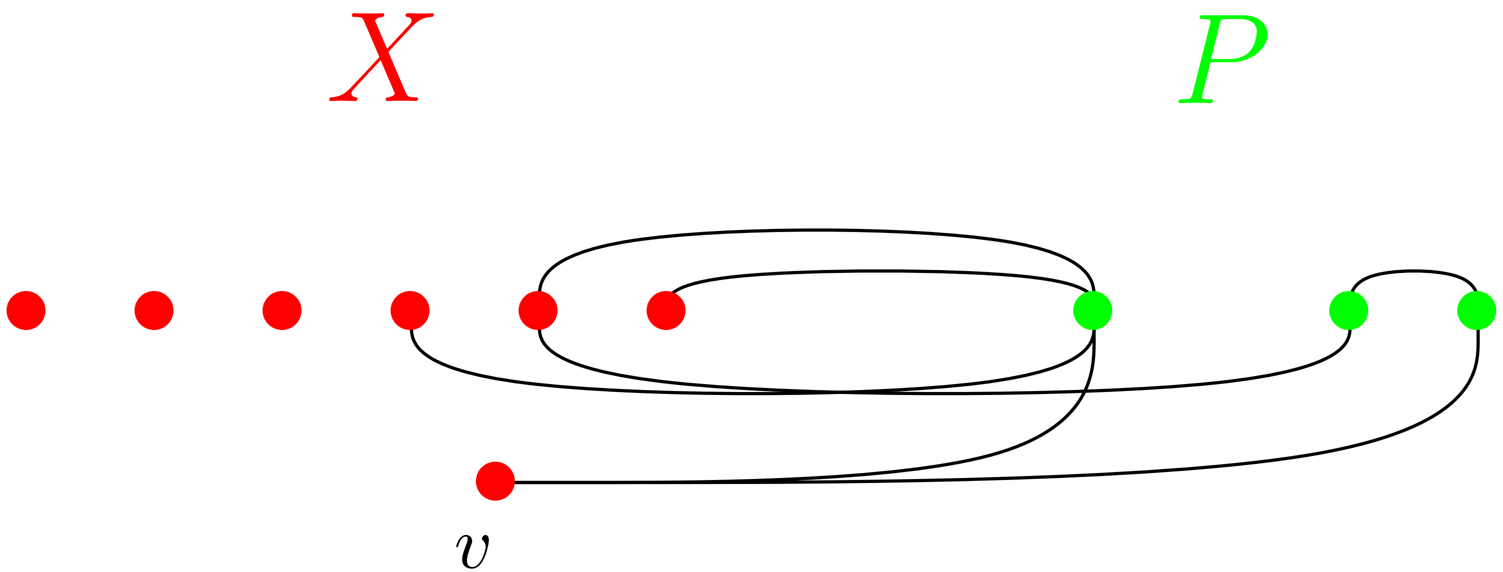

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$$O(|P|^2(|X| + |P|))$$

$$
\begin{aligned}
T(n) &\leq \max_k \{kT(n-k)\} + O(n^2) \\
&= O(3^{n/3})
\end{aligned}
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$$\sum_{v \in V} O(d + |X_v|)3^{d/3})$$

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$$= O(f(d)n) \quad \text{where } f(d) = d3^{d/3}$$

Our running time: $O(dn3^{d/3})$

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Worst-case output size: $O(d(n-d)3^{d/3})$

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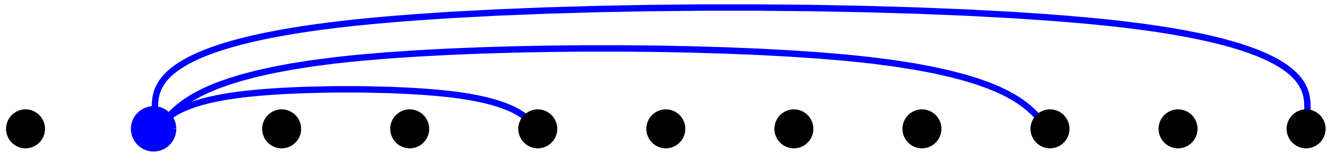
When $n - d = \Omega(n)$, our algorithm is worst-case optimal.

An upper bound

$\leq d$ later neighbors.



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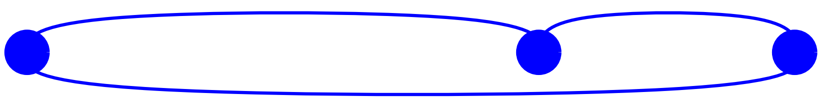

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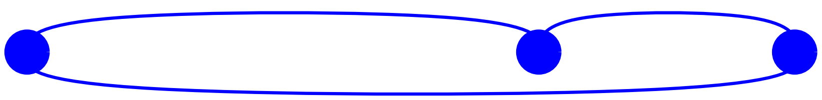

at most $O(3^{d/3})$ maximal cliques

An upper bound

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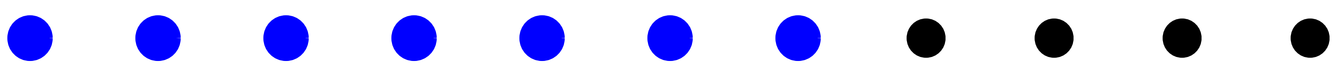
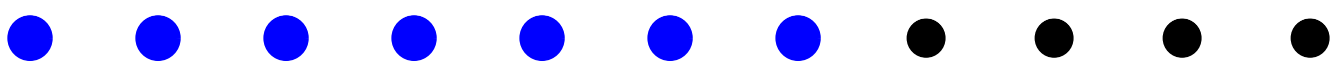
$n-d-3$ vertices $\qquad$ $d+3$ vertices



An upper bound

$(n-d-3)3^{d/3}$ $\qquad$ $3^{\frac{d+3}{3}}$

$n-d-3$ vertices $\qquad$ $d+3$ vertices



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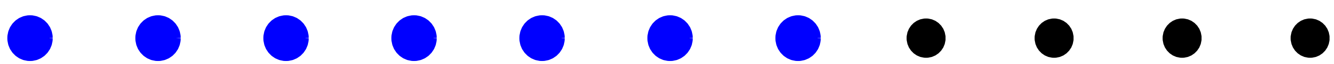
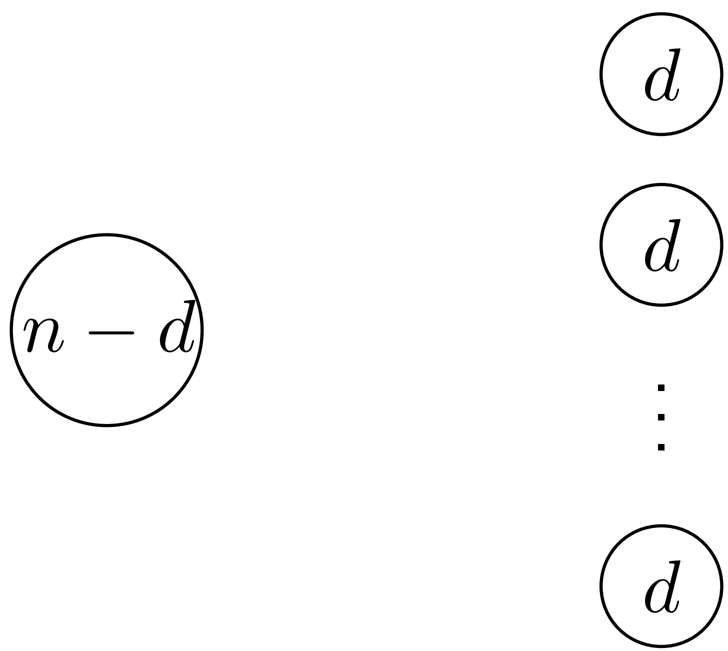


A lower bound

$K_{n-d,3,3,3,\ldots}$

A lower bound

$$K_{n-d,3,3,3,\dots}$$



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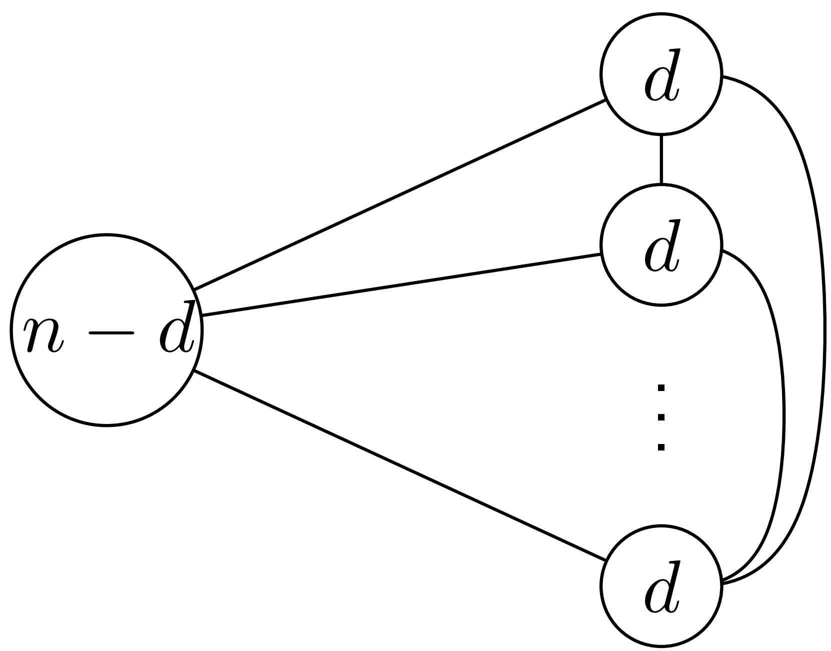

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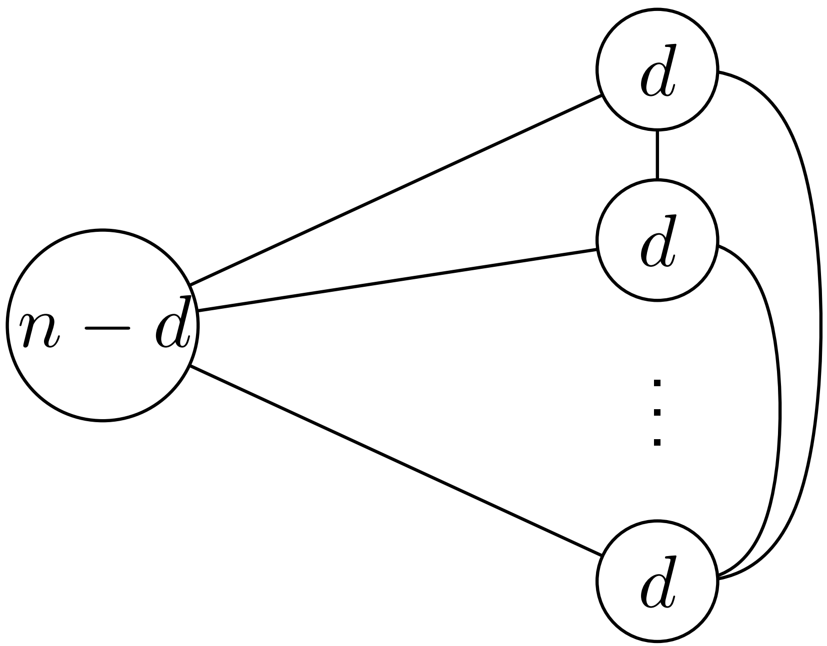

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maximal cliques

A lower bound

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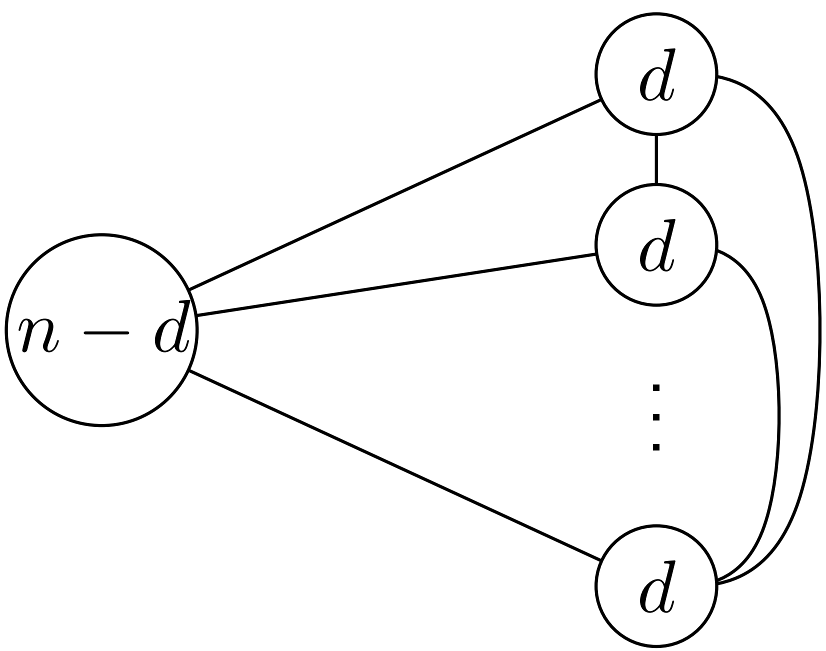

$(n-d)3^{d/3}$
maximal cliques

has degeneracy d
when $(n-d) \geq 3$

at most $(n-d)3^{d/3}$ maximal cliques

each clique is of size at most $d+1$

$O(d(n-d)3^{d/3})$ worst-case output size.

The Bron–Kerbosch Algorithm

proc BronKerbosch(P, R, X)

if $P \cup X = \emptyset$ **then**

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end if

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Thank you!