Cycle Length Distributions in Graphical Models for Iterative Decoding

Xianping Ge
Information & Computer Science
University of California, Irvine
Irvine, CA 92697, U.S.A.
e-mail: `xge@ics.uci.edu`

David Eppstein
Information & Computer Science
University of California, Irvine
Irvine, CA 92697, U.S.A.
e-mail: `eppstein@ics.uci.edu`

Padhraic Smyth
Information & Computer Science
University of California, Irvine
Irvine, CA 92697, U.S.A.
e-mail: `smyth@ics.uci.edu`

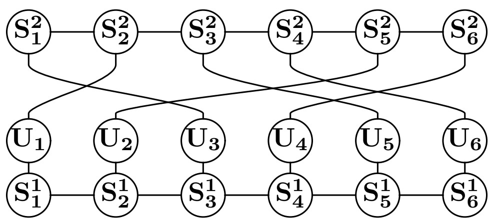

Figure 1: An example of a turbo decoding graph for a $K = 6$, $n = 12$, rate 1/2 turbocode.

I. Introduction

This paper analyzes the distribution of cycle lengths in turbo decoding graphs. It is known that the widely-used iterative decoding algorithm for turbo codes is in fact a special case of a quite general local message-passing algorithm [1] for efficiently computing posterior probabilities in acyclic directed graphical (ADG) models (also known as "belief networks") [2, 3]. However, this local message-passing algorithm in theory only works for graphs with no cycles; why it works in practice (i.e., performs near-optimally in terms of bit decisions) on ADGs for turbo codes is something of a mystery, since turbo decoding graphs can have many cycles. The properties of such cycles are of significant interest in the context of iterative decoding algorithms which are based on belief propagation or message passing.

II. Method

The ADG model for a turbo-decoder can be reduced to what we call a *turbo decoding graph* (Figure 1), which is an undirected graph capturing the inherent loop structure of a turbo decoder. There are two parallel chains, each having n nodes. (For real turbo codes, n can be very large, e.g. $n = 64,000$.) Each node is connected (via a **U** node) to exactly one node on the other chain and these one-to-one connections are chosen randomly, e.g., by a random permutation of the sequence $\{1, 2, \ldots, n\}$.

To help count the cycles in the graph, we drop the **U** nodes, and label the edges in any simple cycle as

1. $\rightarrow$: "Left-to-right on a chain" (e.g., $\mathbf{S_1^2} \rightarrow \mathbf{S_2^2}$ in Figure 1),
2. $\leftarrow$: "Right-to-left on a chain" (e.g., $\mathbf{S_3^1} \leftarrow \mathbf{S_4^1}$), or
3. $=$: "Across the chains" (e.g., $\mathbf{S_3^1} - \mathbf{S_1^2}$).

For example, the cycle $\mathbf{S_1^2} - \mathbf{S_2^2} - \mathbf{S_3^2} - \mathbf{S_5^1} - \mathbf{S_4^1} - \mathbf{S_3^1} - S_1^2$ will be labeled $\rightarrow\rightarrow=\leftarrow\leftarrow=$. Starting from a node on a chain, and a label sequence $L \in \{\rightarrow, \leftarrow, =\}^+$, there is at most one cycle being labeled L. So we count the number of cycles of length k at a node by computing

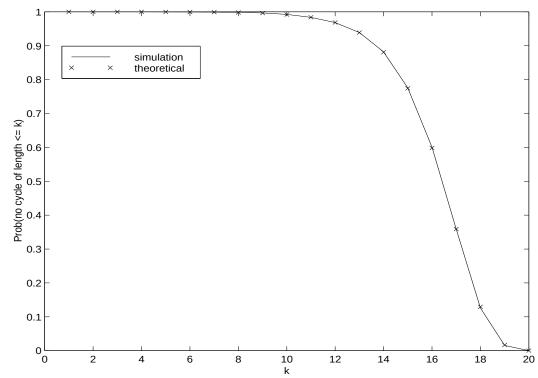

Figure 2: Theoretical vs. simulation estimates of the probability of no cycles of length k or less, as a function of k, in a turbo decoding graph (chain length $n = 64,000$).

1. The total number of possible label sequences L,
2. The probability of finding a cycle with the label sequence L.

See our full paper [4] for details.

III. Conclusions

Using the above method, we estimate the probability that there exist no simple cycles of length $\leq k$ at a randomly chosen node in a turbo decoding graph. In Figure 2, we show the analytical results (together with results from simulations). For turbo codes with a block length of 64000, a randomly chosen node has a less than 1% chance of being on a cycle of length less than or equal to 10, but has a greater than 99.9% chance of being on a cycle of length less than or equal to 20.

References

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